TRIGONOMETRY

TRIGONOMETRIC FUNCTIONS IN A RIGHT-ANGLED TRIANGLE

If A, B, and C are the vertices (C the right angle), and a, b, and h the sides opposite respectively,

$$\text{sine } A = \sin A = \frac{a}{h}, \qquad \text{cosine } A = \cos A = \frac{b}{h},$$

$$\text{tangent } A = \tan A = \frac{a}{b}, \qquad \text{cotangent } A = \cot A = \text{ctn } A = \frac{b}{a},$$

$$\text{secant } A = \sec A = \frac{h}{b}, \qquad \text{cosecant } A = \csc A = \frac{h}{a}.$$

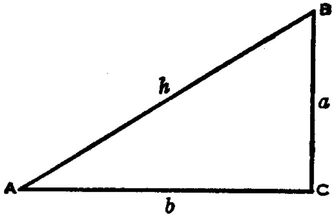


Fig. 4

$$\text{exsecant } A = \text{exsec } A = \sec A - 1$$

$$\text{versine } A = \text{vers } A = 1 - \cos A$$

$$\text{coversine } A = \text{covers } A = 1 - \sin A$$

$$\text{haversine } A = \text{hav } A = \tfrac{1}{2} \text{ vers } A$$

RELATIONS BETWEEN DEGREE OF ACCURACY OF COMPUTED LENGTHS AND ANGLES

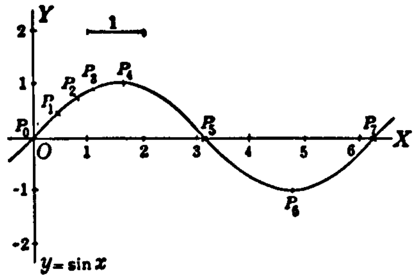
When solving a triangle for any of its parts the following should be observed:

Length to:	(requires)	Angle to:
2 significant digits		nearest $30' = 0.5°$
3 significant digits		nearest $05' = 0.083°$
4 significant digits		nearest $01' = 0.0167°$
5 significant digits		nearest $0.1' = 0.00167°$

SIGNS AND LIMITS OF VALUE ASSUMED BY THE FUNCTIONS

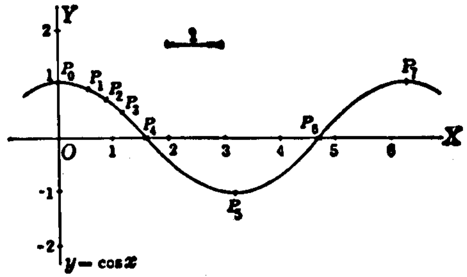
Function	Quadrant I		Quadrant II		Quadrant III		Quadrant IV	
	Sign	Value	Sign	Value	Sign	Value	Sign	Value
sin	+	0 to 1	+	1 to 0	−	0 to 1	−	1 to 0
cos	+	1 to 0	−	0 to 1	−	1 to 0	+	0 to 1
tan	+	0 to ∞	−	∞ to 0	+	0 to ∞	−	∞ to 0
cot	+	∞ to 0	−	0 to ∞	+	∞ to 0	−	0 to ∞
sec	+	1 to ∞	−	∞ to 1	−	1 to ∞	+	∞ to 1
cosec	+	∞ to 1	+	1 to ∞	−	∞ to 1	−	1 to ∞

*GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

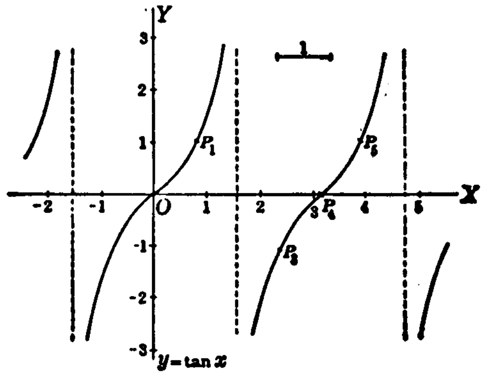


x	y	Point
0	0	$P_0(0, 0)$
$\frac{\pi}{6} = .52$	.50	$P_1(.52, .50)$
$\frac{\pi}{4} = .79$	.71	$P_2(.79, .71)$
$\frac{\pi}{3} = 1.05$	.87	$P_3(1.05, .87)$
$\frac{\pi}{2} = 1.57$	1	$P_4(1.57, 1)$
$\pi = 3.14$	0	$P_5(3.14, 0)$
$\frac{3\pi}{2} = 4.71$	−1	$P_6(4.71, -1)$
$2\pi = 6.28$	0	$P_7(6.28, 0)$

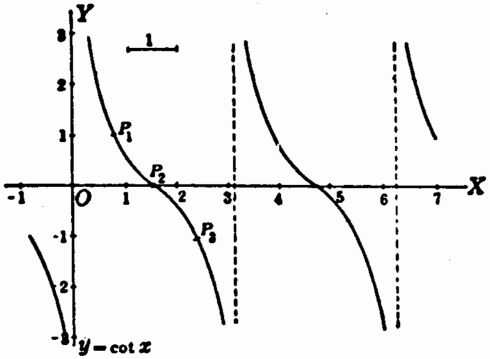
x	y	Point
0	1	$P_0(0, 1)$
$\frac{\pi}{6} = .52$	.87	$P_1(.52, .87)$
$\frac{\pi}{4} = .79$	.71	$P_2(.79, .71)$
$\frac{\pi}{3} = 1.05$	.5	$P_3(1.05, .5)$
$\frac{\pi}{2} = 1.57$	0	$P_4(1.57, 0)$
$\pi = 3.14$	−1	$P_5(3.14, -1)$
$\frac{3\pi}{2} = 4.71$	0	$P_6(4.71, 0)$
$2\pi = 6.28$	1	$P_7(6.28, 1)$



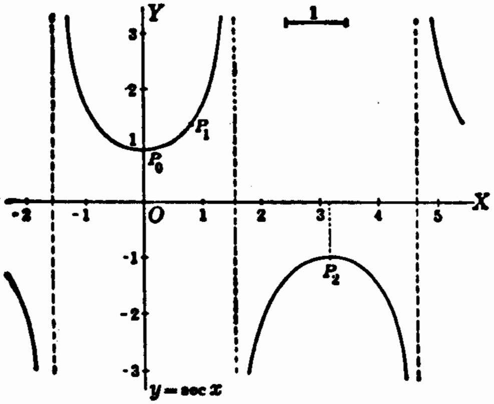
x	y	Point
0	0	$P_0(0, 0)$
$\frac{\pi}{4}$	1	$P_1(.79, 1)$
$\frac{\pi}{2}$	∞	....
$\frac{3\pi}{4}$	−1	$P_3(2.36, -1)$
π	0	$P_4(3.14, 0)$
$\frac{5\pi}{4}$	1	$P_5(3.93, 1)$
$\frac{3\pi}{2}$	∞	....
2π	0	$P_6(6.28, 0)$



GRAPHS OF THE TRIGONOMETRIC FUNCTIONS

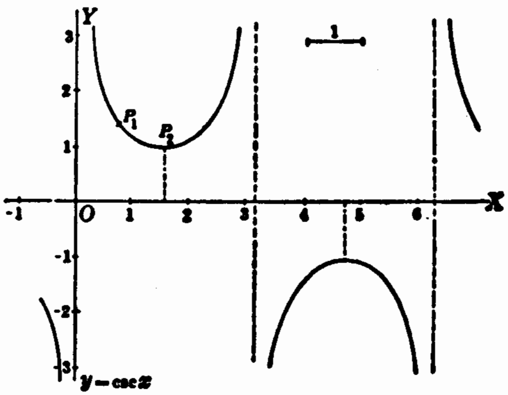


x	y	Point
0	$\pm\infty$	....
$\frac{\pi}{4}$	1	$P_1(.79,\ 1)$
$\frac{\pi}{2}$	0	$P_2(1.57,\ 0)$
$\frac{3\pi}{4}$	-1	$P_3(2.36,\ -1)$
π	$\pm\infty$	....



x	y	Point
0	1	$P_0(0,\ 1)$
$\frac{\pi}{4}$	$\sqrt{2}$	$P_1(.79,\ 1.4)$
$\frac{\pi}{2}$	$\pm\infty$	....
π	-1	$P_2(3.14,\ -1)$

x	y	Point
0	$\pm\infty$	....
$\frac{\pi}{4}$	$\sqrt{2}$	$P_1(.79,\ 1.4)$
$\frac{\pi}{2}$	1	$P_2(1.57,\ 1)$
π	$\pm\infty$	....



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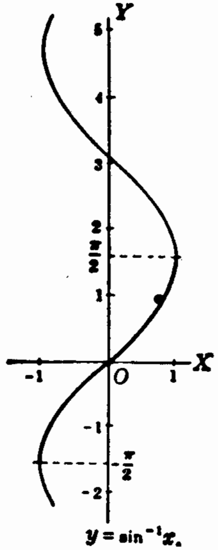
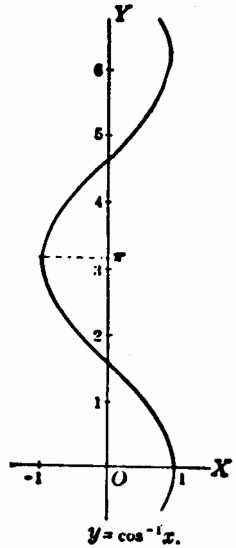




* See index for Table of Curves and Surfaces.

VALUE OF THE FUNCTIONS OF VARIOUS ANGLES

	0°	30°	45°	60°	90°	180°	270°
sin.......	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0	-1
cos.......	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1	0
tan.......	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	∞	0	∞
cot.......	∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0	∞	0
sec........	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-1	∞
cosec......	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	∞	-1

EXPONENTIAL DEFINITIONS OF CIRCULAR FUNCTIONS

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \qquad \operatorname{cosec} x = \frac{2i}{e^{ix} - e^{-ix}}$$

$(i^2 = -1)$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sec x = \frac{2}{e^{ix} + e^{-ix}}$$

$$\tan x = \frac{e^{ix} - e^{-ix}}{ie^{ix} + ie^{-ix}} \qquad \cot x = \frac{ie^{ix} + ie^{-ix}}{e^{ix} - e^{-ix}}$$

RELATIONS OF THE FUNCTIONS

$$\sin x = \frac{1}{\operatorname{cosec} x}. \qquad \operatorname{cosec} x = \frac{1}{\sin x}.$$

$$\cos x = \frac{1}{\sec x}. \qquad \sec x = \frac{1}{\cos x}.$$

$$\tan x = \frac{1}{\cot x} = \frac{\sin x}{\cos x}. \qquad \sin^2 x + \cos^2 x = 1.$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}. \qquad 1 + \tan^2 x = \sec^2 x.$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x.$$

$*\sin x = \pm\sqrt{1 - \cos^2 x}.$ $\qquad$ $*\cos x = \pm\sqrt{1 - \sin^2 x}.$

$*\tan x = \pm\sqrt{\sec^2 x - 1}.$ $\qquad$ $*\sec x = \pm\sqrt{\tan^2 x + 1}.$

$*\cot x = \pm\sqrt{\operatorname{cosec}^2 x - 1}.$ $\qquad$ $*\operatorname{cosec} x = \pm\sqrt{\cot^2 x + 1}.$

$\sin x = \cos(90° - x) = \sin(180° - x).$

$\cos x = \sin(90° - x) = -\cos(180° - x).$

$\tan x = \cot(90° - x) = -\tan(180° - x).$

$\cot x = \tan(90° - x) = -\cot(180° - x).$

$$\operatorname{cosec} x = \cot\frac{x}{2} - \cot x.$$

* The sign in front of radical depends on quadrant in which x falls.

FUNCTIONS OF SUMS OF ANGLES

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y.$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

FUNCTIONS OF MULTIPLE ANGLES

$\sin 2x = 2 \sin x \cos x.$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$

$\sin 3x = 3 \sin x - 4 \sin^3 x.$

$\cos 3x = 4 \cos^3 x - 3 \cos x.$

$\sin 4x = 8 \cos^3 x \sin x - 4 \cos x \sin x.$

$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.$

$\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x.$

$\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x.$

$\sin 6x = 32 \cos^5 x \sin x - 32 \cos^3 x \sin x + 6 \cos x \sin x.$

$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1.$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

$$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

$$*\sin \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$*\cos \tfrac{1}{2}x = \pm\sqrt{\frac{1 + \cos x}{2}}.$$

$$*\tan \tfrac{1}{2}x = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}.$$

* The sign in front of radical depends on quadrant in which $\frac{x}{2}$ falls.

MISCELLANEOUS RELATIONS

$\sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cdot \cos \frac{1}{2}(x \mp y).$

$\cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cdot \cos \frac{1}{2}(x - y).$

$\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \cdot \sin \frac{1}{2}(x - y).$

$\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$

$\cos x \sin y = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$

$\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cdot \cos y}. \quad \cot x \pm \cot y = \frac{\pm \sin(x \pm y)}{\sin x \cdot \sin y}.$$

$$\frac{1 + \tan x}{1 - \tan x} = \tan(45° + x). \quad \frac{\cot x + 1}{\cot x - 1} = \cot(45° - x).$$

$$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \tfrac{1}{2}(x \pm y).$$

$$\frac{\sin x \pm \sin y}{\cos x - \cos y} = -\cot \tfrac{1}{2}(x \mp y).$$

$$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\tan \frac{1}{2}(x + y)}{\tan \frac{1}{2}(x - y)}.$$

$\sin^2 x - \sin^2 y = \sin(x + y) \cdot \sin(x - y).$

$\cos^2 x - \cos^2 y = -\sin(x + y) \sin(x - y).$

$\cos^2 x - \sin^2 y = \cos(x + y) \cos(x - y).$

INVERSE TRIGONOMETRIC FUNCTIONS

The following table lists each of the six inverse trigonometric functions together with the interval of its principal value:

Function	Interval containing principal value	
	x positive or zero	x negative
$y = \sin^{-1} x$ and $\tan^{-1} x$	$0 \leqq y \leqq \pi/2$	$-\pi/2 \leqq y < 0$
$y = \cos^{-1} x$ and $\cot^{-1} x$	$0 \leqq y \leqq \pi/2$	$\pi/2 < y \leqq \pi$
$y = \sec^{-1} x$ and $\csc^{-1} x$	$0 \leqq y \leqq \pi/2$	$-\pi \leqq y \leqq -\pi/2$

Usually the first letter in "arc" or the name of the inverse trigonometric functions is capitalized if the principal value is desired. Thus

$$\text{Arc sin} \frac{1}{2} = \text{Sin}^{-1} \frac{1}{2} = \frac{\pi}{6},$$

while

$$\text{arc sin} \frac{1}{2} = \frac{\pi}{6} + 2\pi n \text{ or } \frac{5\pi}{6} + 2\pi n$$

In the calculus both for differentiation and integral formulas, capitalization is not adhered to strictly, but principal values are always understood for inverse trigonometric functions when used unless specifically stated otherwise.

RELATIONS BETWEEN SIDES AND ANGLES OF ANY PLANE TRIANGLE

In a triangle with angles A, B, and C and sides opposite a, b, and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter of the circumscribed circle.}$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$a = b \cos C + c \cos B.$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}.$$

$$\sin A = \frac{2}{bc}\sqrt{s(s - a)(s - b)(s - c)},$$

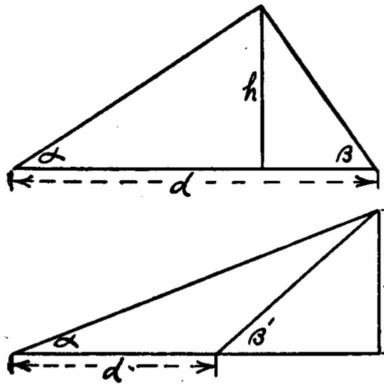
where $s = \frac{1}{2}(a + b + c)$ and $r = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}.$

$$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}}.$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} = \frac{r}{s - a}.$$

$$\frac{a + b}{a - b} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(A - B)}.$$



$$h = d\,\frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)} = \frac{d}{\cot \alpha + \cot \beta}$$

Similarly

$$h = d\,\frac{\sin \alpha \sin \beta'}{\sin(\beta' - \alpha)} = \frac{d}{\cot \alpha - \cot \beta'}$$

Fig. 5

$d\,(au) = a\,du$

$d\,(u + v - w) = du + dv - dw$

$d\,(uv) = u\,dv + v\,du$

$d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$

$d(u^n) = nu^{n-1}\,du$

$d(u^v) = vu^{v-1}\,du + u^v(\log_e u)\,dv$

$d\,(e^u) = e^u\,du$

$d(e^{au}) = ae^{au}\,du$

$d\,(a^u) = a^u(\log_e a)\,du$

$d\,(\log_e u) = u^{-1}du$

$d\,(\log_a u) = u^{-1}(\log_a e)\,du$

$d\,(u^u) = u^u(1 + \log_e u)\,du$

$d \sin u = \cos u\,du$

$d \cos u = -\sin u\,du$

$d \tan u = \sec^2 u\,du$

$d \cot u = -\csc^2 u\,du$

$d \sec u = \tan u \sec u\,du$

$d \csc u = -\cot u \cdot \csc u\,du$

$d\,\text{vers}\,u = \sin u\,du$

$d \sin^{-1} u = (1 - u^2)^{-\frac{1}{2}}\,du$

$d \cos^{-1} u = -(1 - u^2)^{-\frac{1}{2}}du$

$d \tan^{-1} u = (1 + u^2)^{-1}du$

$d \cot^{-1} u = -(1 + u^2)^{-1}du$

$d \sec^{-1} u = u^{-1}(u^2 - 1)^{-\frac{1}{2}}du$

$d \csc^{-1} u = -u^{-1}(u^2 - 1)^{-\frac{1}{2}}du$

$d \sinh u = \cosh u\,du$

$d \cosh u = \sinh u\,du$

$d \tanh u = \text{sech}^2\,u\,du$

$d \coth u = -\text{csch}^2\,u\,du$

$d\,\text{sech}\,u = -\text{sech}\,u \tanh u\,du$

$d\,\text{csch}\,u = -\text{csch}\,u \coth u\,du$

$d\,\text{vers}^{-1}\,u = (2u - u^2)^{-\frac{1}{2}}\,du$

$d \sinh^{-1} u = (u^2 + 1)^{-\frac{1}{2}}du$

$d \cosh^{-1} u = (u^2 - 1)^{-\frac{1}{2}}\,du$

$d \tanh^{-1} u = (1 - u^2)^{-1}\,du$

$d \coth^{-1} u = -(u^2 - 1)^{-1}du$

$d\,\text{sech}^{-1}\,u = -u^{-1}(1 - u^2)^{-\frac{1}{2}}\,du$

$d\,\text{csch}^{-1}\,u = -u^{-1}(u^2 + 1)^{-\frac{1}{2}}du$