TABLE OF INTEGRALS

The following points should be observed when using this table.

1. A constant of integration is to be supplied with the answers for indefinite integrals.

2. Logarithmic expressions are to base $e = 2.71828 \cdots$, unless otherwise specified, and are to be evaluated for the absolute value of the arguments involved therein.

*3. All angles are measured in radians, and inverse trigonometric functions represent principal angles.

4. If the application of a formula produces either a zero denominator or a radical involving the unit $i = \sqrt{-1}$ in the result, there is always available another form of the answer which avoids this difficulty. In many of the results, the excluded values are specified, but when such are omitted it is presumed that one can tell what these should be, especially when difficulties of the type herein mentioned are obtained.

* See index for table Inverse Trigonometric Functions.

ELEMENTARY FORMS

1. $\int a\,dx = ax.$

2. $\int a \cdot f(x)dx = a\int f(x)dx.$

3. $\int \phi(y)dx = \int \frac{\phi(y)}{y'}\,dy,$ where $y' = dy/dx.$

4. $\int (u + v)\,dx = \int u\,dx + \int v\,dx,$ where u and v are any functions of x.

*__5.__ $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du.$

6. $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$

7. $\int x^n\,dx = \frac{x^{n+1}}{n+1},$ except $n = -1.$

8. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x),$ $[d\,f(x) = f'(x)\,dx].$

9. $\int \frac{dx}{x} = \log x,$ or $\log(-x).$

10. $\int \frac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)},$ $[d\,f(x) = f'(x)\,dx].$

* See index for table "Special Integration Formulas" for formula on "extended rule for integration by parts."

11. $\int e^x\,dx = e^x.$

12. $\int e^{ax}\,dx = e^{ax}/a.$

13. $\int b^{ax}\,dx = \dfrac{b^{ax}}{a \log b}.$

14. $\int \log x\,dx = x \log x - x.$

15. $\int a^x \log a\,dx = a^x.$

16. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$, or $-\dfrac{1}{a}\cot^{-1}\left(\dfrac{x}{a}\right).$

17. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right)$, or $\dfrac{1}{2a}\log\dfrac{a+x}{a-x}.$

18. $\int \dfrac{dx}{x^2 - a^2} = -\dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right)$, or $\dfrac{1}{2a}\log\dfrac{x-a}{x+a}.$

19. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$, or $-\cos^{-1}\left(\dfrac{x}{a}\right).$

20. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \log\,(x + \sqrt{x^2 \pm a^2}).$

21. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left(\dfrac{a}{x}\right).$

22. $\int \dfrac{dx}{x\sqrt{a^2 \pm x^2}} = -\dfrac{1}{a}\log\left(\dfrac{a + \sqrt{a^2 \pm x^2}}{x}\right).$

23. $\int \dfrac{dx}{x\sqrt{a + bx}} = \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bx}{-a}}$, or

$\dfrac{-2}{\sqrt{a}}\tanh^{-1}\sqrt{\dfrac{a+bx}{a}}$ or $\dfrac{1}{\sqrt{a}}\log\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}.$

Forms Containing $(a + bx)$

24. $\int (a + bx)^n dx = \dfrac{(a+bx)^{n+1}}{(n+1)b}$, except $n = -1.$

25. $\int x\,(a + bx)^n dx = \dfrac{1}{b^2(n+2)}\,(a+bx)^{n+2}$

$-\dfrac{a}{b^2(n+1)}\,(a+bx)^{n+1}$, except $n = -1$ or $-2.$

26. $\int x^2\,(a + bx)^n\,dx = \dfrac{1}{b^3}\left[\dfrac{(a+bx)^{n+3}}{n+3} - 2a\,\dfrac{(a+bx)^{n+2}}{n+2} + a^2\,\dfrac{(a+bx)^{n+1}}{n+1}\right]$

27. $\int x^m(a + bx)^n\,dx =$

$\dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\int x^m(a+bx)^{n-1}\,dx.$

28. $\int x^m\,(a + bx)^n\,dx = \dfrac{1}{a(n+1)}\left[-x^{m+1}(a+bx)^{n+1} + (m+n+2)\int x^m(a+bx)^{n+1}\,dx\right].$

29. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\log\,(a+bx).$

30. $\int \dfrac{dx}{(a+bx)^2} = -\dfrac{1}{b\,(a+bx)}.$

31. $\int \dfrac{dx}{(a+bx)^3} = -\dfrac{1}{2\,b\,(a+bx)^2}.$

32. $\int \dfrac{xdx}{a+bx} = \dfrac{1}{b^2}\,[a + bx - a\log\,(a+bx)].$

33. $\int \dfrac{xdx}{(a+bx)^2} = \dfrac{1}{b^2}\left[\log\,(a+bx) + \dfrac{a}{a+bx}\right].$

34. $\int \dfrac{xdx}{(a+bx)^3} = \dfrac{1}{b^2}\left[-\dfrac{1}{a+bx} + \dfrac{a}{2(a+bx)^2}\right].$

35. $\int \dfrac{xdx}{(a+bx)^n} = \dfrac{1}{b^2}\left[\dfrac{-1}{(n-2)(a+bx)^{n-2}} + \dfrac{a}{(n-1)(a+bx)^{n-1}}\right], n \neq 1, 2.$

36. $\int \dfrac{x^2dx}{a+bx} = \dfrac{1}{b^3}\left[\dfrac{1}{2}\,(a+bx)^2 - 2\,a\,(a+bx) + a^2\log\,(a+bx)\right].$

37. $\int \dfrac{x^2\,dx}{(a+bx)^2} = \dfrac{1}{b^3}\left[a + bx - 2\,a\log\,(a+bx) - \dfrac{a^2}{a+bx}\right].$

38. $\int \frac{x^2\,dx}{(a+bx)^3} = \frac{1}{b^3}\left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$.

39. $\int \frac{x^2\,dx}{(a+bx)^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right], n \neq 1, 2, 3.$

40. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$.

41. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$.

42. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$.

43. $\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}$.

Forms Containing $c^2 \pm x^2$, $x^2 - c^2$

44. $\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}$, or $\frac{1}{c}\sin^{-1}\frac{x}{\sqrt{c^2+x^2}}$.

45. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}$, or $\frac{1}{c}\tanh^{-1}\left(\frac{x}{c}\right)$.

46. $\int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}$, or $-\frac{1}{c}\coth^{-1}\left(\frac{x}{c}\right)$.

Forms Containing $a + bx$ and $a' + b'x$

47. $\int \frac{dx}{(a+bx)(a'+b'x)} = \frac{1}{ab'-a'b}\cdot\log\left(\frac{a'+b'x}{a+bx}\right)$.

48. $\int \frac{x\,dx}{(a+bx)(a'+b'x)} = \frac{1}{ab'-a'b}\left[\frac{a}{b}\log(a+bx) - \frac{a'}{b'}\log(a'+b'x)\right]$.

49. $\int \frac{dx}{(a+bx)^2(a'+b'x)} = \frac{1}{ab'-a'b}\left(\frac{1}{a+bx} + \frac{b'}{ab'-a'b}\log\frac{a'+b'x}{a+bx}\right)$.

50. $\int \frac{x\,dx}{(a+bx)^2(a'+b'x)} = \frac{-a}{b(ab'-a'b)(a+bx)} - \frac{a'}{(ab'-a'b)^2}\log\frac{a'+b'x}{a+bx}$.

51. $\int \frac{x^2\,dx}{(a+bx)^2(a'+b'x)} = \frac{a^2}{b^2(ab'-a'b)(a+bx)} + \frac{1}{(ab'-a'b)^2}\left[\frac{a'^2}{b'}\log(a'+b'x) + \frac{a(ab'-2a'b)}{b^2}\log(a+bx)\right]$.

52. $\int \frac{dx}{(a+bx)^n(a'+b'x)^m} = \frac{1}{(m-1)(ab'-a'b)}\left(\frac{-1}{(a+bx)^{n-1}(a'+b'x)^{m-1}} - (m+n-2)b\int\frac{dx}{(a+bx)^n(a'+b'x)^{m-1}}\right)$.

53. $\int \frac{a+bx}{a'+b'x}\,dx = \frac{bx}{b'} + \frac{ab'-a'b}{b'^2}\log(a'+b'x)$.

54. $\int \frac{(a+bx)^m dx}{(a'+b'x)^n} = -\frac{1}{(n-1)(ab'-a'b)}\left[\frac{(a+bx)^{m+1}}{(a'+b'x)^{n-1}} + b(n-m-2)\int\frac{(a+bx)^m\,dx}{(a'+b'x)^{n-1}}\right]$

$= -\frac{1}{b'(n-m-1)}\left[\frac{(a+bx)^m}{(a'+b'x)^{n-1}} + m(ab'-a'b)\int\frac{(a+bx)^{m-1}dx}{(a'+b'x)^n}\right]$

$= \frac{-1}{(n-1)b'}\left[\frac{(a+bx)^m}{(a'+b'x)^{n-1}} - mb\int\frac{(a+bx)^{m-1}dx}{(a'+b'x)^{n-1}}\right]$.

Forms Containing $\sqrt{a+bx} = \sqrt{u}$ and $\sqrt{a'+b'x} = \sqrt{v}$ with $k = ab' - a'b$

55. $\int \sqrt{uv}\,dx = \frac{k+2bv}{4bb'}\sqrt{uv} - \frac{k^2}{8bb'}\int\frac{dx}{\sqrt{uv}}$.

56. $\int \frac{dx}{v\sqrt{u}} = \frac{1}{\sqrt{kb'}}\log\frac{b'\sqrt{u}-\sqrt{kb'}}{b'\sqrt{u}+\sqrt{kb'}}$, or

$= \frac{2}{\sqrt{-kb'}}\tan^{-1}\frac{b'\sqrt{u}}{\sqrt{-kb'}}$.

57. $\displaystyle\int \frac{dx}{\sqrt{uv}} = \frac{2}{\sqrt{bb'}} \log\left(\sqrt{bb'u} + b\sqrt{v}\right)$, or

$$= \frac{2}{\sqrt{bb'}} \tanh^{-1} \sqrt{\frac{b'u}{bv}}, \text{ or}$$

$$= \frac{2}{\sqrt{-bb'}} \tan^{-1} \sqrt{\frac{-b'u}{bv}}, \text{ or } \frac{-1}{\sqrt{-bb'}} \sin^{-1} \frac{2bb'x + a'b + ab'}{|k|}.$$

58. $\displaystyle\int \frac{xdx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bb'} - \frac{ab' + a'b}{2bb'} \int \frac{dx}{\sqrt{uv}}.$

59. $\displaystyle\int \frac{dx}{v\sqrt{uv}} = -\frac{2\sqrt{u}}{k\sqrt{v}}.$

60. $\displaystyle\int \frac{\sqrt{v}\,dx}{\sqrt{u}} = \frac{1}{b}\sqrt{uv} - \frac{k}{2b}\int \frac{dx}{\sqrt{uv}}.$

61. $\displaystyle\int v^m \sqrt{u}\,dx = \frac{1}{(2m+3)b'}\left(2v^{m+1}\sqrt{u} + k\int \frac{v^m dx}{\sqrt{u}}\right).$

62. $\displaystyle\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left(\frac{\sqrt{u}}{v^{m-1}} + \left(m - \frac{3}{2}\right) b\int \frac{dx}{v^{m-1}\sqrt{u}}\right).$

Forms Containing $(a + bx^n)$

63. $\displaystyle\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} \frac{x\sqrt{ab}}{a}.$

64. $\displaystyle\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ab}} \log \frac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}$, or

$$\frac{1}{\sqrt{-ab}} \tanh^{-1} \frac{x\sqrt{-ab}}{a}.$$

65. $\displaystyle\int \frac{xdx}{a+bx^2} = \frac{1}{2b} \log\left(x^2 + \frac{a}{b}\right).$

66. $\displaystyle\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+bx^2}.$

67. $\displaystyle\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}.$

68. $\displaystyle\int \frac{dx}{(a+bx^2)^{m+1}} = \frac{1}{2ma}\frac{x}{(a+bx^2)^m} + \frac{2m-1}{2ma}\int \frac{dx}{(a+bx^2)^m}.$

69. $\displaystyle\int \frac{xdx}{(a+bx^2)^{m+1}} = \frac{1}{2}\int \frac{dz}{(a+bz)^{m+1}}, \qquad [z = x^2].$

70. $\displaystyle\int \frac{x^2dx}{(a+bx^2)^{m+1}} = \frac{-x}{2mb(a+bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a+bx^2)^m}.$

71. $\displaystyle\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a}\log\frac{x^2}{a+bx^2}.$

72. $\displaystyle\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}.$

73. $\displaystyle\int \frac{dx}{x(a+bx^2)^{m+1}} = \frac{1}{2am(a+bx^2)^m} + \frac{1}{a}\int \frac{dx}{x(a+bx^2)^m}$

$m \neq 0.$

74. $\displaystyle\int \frac{dx}{x^2(a+bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a+bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a+bx^2)^{m+1}}.$

75. $\displaystyle\int \frac{dx}{a+bx^3} =$

$$\frac{k}{3a}\left[\frac{1}{2}\log\frac{(k+x)^2}{k^2 - kx + x^2} + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right], \quad [bk^3 = a].$$

76. $\displaystyle\int \frac{xdx}{a+bx^3} =$

$$\frac{1}{3bk}\left[\frac{1}{2}\log\frac{k^2 - kx + x^2}{(k+x)^2} + \sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right], \quad [bk^3 = a].$$

77. $\displaystyle\int \frac{dx}{x(a+bx^n)} = \frac{1}{an}\log\frac{x^n}{a+bx^n}.$

78. $\displaystyle\int \frac{dx}{(a+bx^n)^{m+1}} = \frac{1}{a}\int \frac{dx}{(a+bx^n)^m} - \frac{b}{a}\int \frac{x^n dx}{(a+bx^n)^{m+1}}.$

79. $\displaystyle\int \frac{x^m dx}{(a+bx^n)^{p+1}} = \frac{1}{b}\int \frac{x^{m-n}dx}{(a+bx^n)^p} - \frac{a}{b}\int \frac{x^{m-n}dx}{(a+bx^n)^{p+1}}.$

80. $\displaystyle\int \frac{dx}{x^m(a+bx^n)^{p+1}} =$

$$\frac{1}{a}\int \frac{dx}{x^m(a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n}(a+bx^n)^{p+1}}.$$

81. $\displaystyle\int x^m(a+bx^n)^p dx =$

$$\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)} - \frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

82. $\displaystyle\int x^m(a+bx^n)^p\,dx =$

$$\frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx.$$

83. $\int x^{m-1}(a+bx^n)^p\,dx = \frac{1}{b(m+np)}$
$\left[x^{m-n}(a+bx^n)^{p+1} - (m-n)a\int x^{m-n-1}(a+bx^n)^p\,dx\right].$

84. $\int x^{m-1}(a+bx^n)^p\,dx =$
$\frac{1}{m+np}\left[x^m(a+bx^n)^p + npa\int x^{m-1}(a+bx^n)^{p-1}\,dx\right].$

85. $\int x^{m-1}(a+bx^n)^p\,dx =$
$\frac{1}{ma}\left[x^m(a+bx^n)^{p+1} - (m+np+n)b\int x^{m+n-1}(a+bx^n)^p\,dx\right].$

86. $\int x^{m-1}(a+bx^n)^p\,dx = \frac{1}{an(p+1)}\left[-x^m(a+bx^n)^{p+1}\right.$
$\left. + (m+np+n)\int x^{m-1}(a+bx^n)^{p+1}\,dx\right].$

Forms Containing $(a+bx+cx^2)$

$X = a+bx+cx^2$ and $q = 4ac - b^2$

87. $\int \frac{dx}{X} = \frac{2}{\sqrt{q}}\tan^{-1}\frac{2cx+b}{\sqrt{q}}.$

88. $\int \frac{dx}{X} = \frac{-2}{\sqrt{-q}}\tanh^{-1}\frac{2cx+b}{\sqrt{-q}}.$

89. $\int \frac{dx}{X} = \frac{1}{\sqrt{-q}}\log\frac{2cx+b-\sqrt{-q}}{2cx+b+\sqrt{-q}}.$

90. $\int \frac{dx}{X^2} = \frac{2cx+b}{qX} + \frac{2c}{q}\int\frac{dx}{X}.$

91. $\int \frac{dx}{X^3} = \frac{2cx+b}{q}\left(\frac{1}{2X^2} + \frac{3c}{qX}\right) + \frac{6c^2}{q^2}\int\frac{dx}{X}.$

92. $\int \frac{dx}{X^{n+1}} = \frac{2cx+b}{nqX^n} + \frac{2(2n-1)c}{qn}\int\frac{dx}{X^n}.$

93. $\int \frac{xdx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int\frac{dx}{X}.$

94. $\int \frac{xdx}{X^2} = -\frac{bx+2a}{qX} - \frac{b}{q}\int\frac{dx}{X}.$

95. $\int \frac{xdx}{X^{n+1}} = -\frac{2a+bx}{nqX^n} - \frac{b(2n-1)}{nq}\int\frac{dx}{X^n}.$

96. $\int \frac{x^2}{X}\,dx = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int\frac{dx}{X}.$

97. $\int \frac{x^2}{X^2}\,dx = \frac{(b^2-2ac)x+ab}{cqX} + \frac{2a}{q}\int\frac{dx}{X}.$

98. $\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n-m+1)cX^n}$
$-\frac{n-m+1}{2n-m+1}\cdot\frac{b}{c}\int\frac{x^{m-1}\,dx}{X^{n+1}} + \frac{m-1}{2n-m+1}\cdot\frac{a}{c}\int\frac{x^{m-2}\,dx}{X^{n+1}}.$

99. $\int \frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int\frac{dx}{X}.$

100. $\int \frac{dx}{x^2X} = \frac{b}{2a^2}\log\frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right)\int\frac{dx}{X}.$

101. $\int \frac{dx}{xX^n} = \frac{1}{2a(n-1)X^{n-1}} - \frac{b}{2a}\int\frac{dx}{X^n} + \frac{1}{a}\int\frac{dx}{xX^{n-1}}.$

102. $\int \frac{dx}{x^mX^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\cdot$
$\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}} - \frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}}.$

Forms Containing $\sqrt{a+bx}$

103. $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$

104. $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

105. $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$

106. $\int x^m\sqrt{a+bx}\,dx =$
$\frac{2}{b(2m+3)}\left[x^m\sqrt{(a+bx)^3} - ma\int x^{m-1}\sqrt{a+bx}\,dx\right].$

107. $\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$
(see No. 114 and No. 115).

108. $\int \frac{\sqrt{a+bx}}{x^2}\,dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{a+bx}}.$
(see No. 114 and No. 115).

109. $$\int \frac{\sqrt{a+bx}}{x^m} = -\frac{1}{(m-1)a}\left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2}\int \frac{\sqrt{a+bx}\,dx}{x^{m-1}}\right] \quad m \neq 1.$$

110. $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$$

111. $$\int \frac{xdx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}.$$

112. $$\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}.$$

113. $$\int \frac{x^m dx}{\sqrt{a+bx}} = \frac{2x^m\sqrt{a+bx}}{(2m+1)b} - \frac{2ma}{(2m+1)b}\int \frac{x^{m-1}dx}{\sqrt{a+bx}}.$$

114. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right). \quad a>0$$ See #23 for $a<0$

115. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{-2}{\sqrt{a}}\tanh^{-1}\sqrt{\frac{a+bx}{a}}. \quad a>0$$

116. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}.$$

117. $$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \frac{(2n-3)b}{(2n-2)a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}.$$

118. $$\int (a+bx)^{\pm n/2}dx = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}.$$

119. $$\int x(a+bx)^{\pm n/2}dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right].$$

120. $$\int \frac{dx}{x(a+bx)^{m/2}} = \frac{1}{a}\int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{m/2}}.$$

121. $$\int \frac{(a+bx)^{n/2}dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}dx.$$

122. $$\int f(x,\sqrt{a+bx})\,dx = \frac{2}{b}\int f\left(\frac{z^2-a}{b}, z\right)zdz \qquad (z^2 = a+bx).$$

Forms Containing $\sqrt{x^2 \pm a^2}$

123. $$\int \sqrt{x^2\pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2\pm a^2} \pm a^2\log(x+\sqrt{x^2\pm a^2})].$$

124. $$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \log(x+\sqrt{x^2\pm a^2}).$$

125. $$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)$$

126. $$\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$$

127. $$\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} - a\log\left(\frac{a+\sqrt{x^2+a^2}}{x}\right).$$

128. $$\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\sec^{-1}\frac{a}{x}.$$

129. $$\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2}.$$

130. $$\int x\sqrt{x^2\pm a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2\pm a^2)^3}.$$

131. $$\int \sqrt{(x^2\pm a^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(x^2\pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2\pm a^2} + \frac{3a^4}{2}\log(x+\sqrt{x^2\pm a^2})\right].$$

132. $$\int \frac{dx}{\sqrt{(x^2\pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}.$$

133. $$\int \frac{x\,dx}{\sqrt{(x^2\pm a^2)^3}} = \frac{-1}{\sqrt{x^2\pm a^2}}.$$

134. $$\int x\sqrt{(x^2\pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2\pm a^2)^5}.$$

135. $$\int x^2\sqrt{x^2\pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2\pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2\pm a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2\pm a^2}).$$

136. $$\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2)\sqrt{(a^2+x^2)^3}.$$

137. $\int x^3\sqrt{x^2-a^2}\,dx = \frac{1}{5}\sqrt{(x^2-a^2)^5} + \frac{a^2}{3}\sqrt{(x^2-a^2)^3}.$

138. $\int \frac{x^2dx}{\sqrt{x^2\pm a^2}} = \frac{x}{2}\sqrt{x^2\pm a^2} \mp \frac{a^2}{2}\log\,(x+\sqrt{x^2\pm a^2}).$

139. $\int \frac{x^3dx}{\sqrt{x^2\pm a^2}} = \frac{1}{3}\sqrt{(x^2\pm a^2)^3} \mp a^2\sqrt{x^2\pm a^2}.$

140. $\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}.$

141. $\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}.$

142. $\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\sec^{-1}\frac{a}{x}.$

143. $\int x^2\sqrt{(x^2\pm a^2)^3}\,dx = \frac{x}{6}\sqrt{(x^2\pm a^2)^5}$

$$\mp\frac{a^2x}{24}\sqrt{(x^2\pm a^2)^3} - \frac{a^4x}{16}\sqrt{x^2\pm a^2} \mp \frac{a^6}{16}\log\,(x+\sqrt{x^2\pm a^2}).$$

144. $\int x^3\sqrt{(x^2\pm a^2)^3}\,dx = \frac{1}{7}\sqrt{(x^2\pm a^2)^7} \mp \frac{a^2}{5}\sqrt{(x^2\pm a^2)^5}.$

145. $\int \frac{\sqrt{x^2\pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2\pm a^2}}{x} + \log\,(x+\sqrt{x^2\pm a^2}).$

146. $\int \frac{\sqrt{x^2+a^2}}{x^3}dx = -\frac{\sqrt{x^2+a^2}}{2x^2} - \frac{1}{2a}\log\frac{a+\sqrt{x^2+a^2}}{x}.$

147. $\int \frac{\sqrt{x^2-a^2}}{x^3}dx = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\sec^{-1}\frac{a}{x}.$

148. $\int \frac{x^2dx}{\sqrt{(x^2\pm a^2)^3}} = \frac{-x}{\sqrt{x^2\pm a^2}} + \log\,(x+\sqrt{x^2\pm a^2}).$

149. $\int \frac{x^3dx}{\sqrt{(x^2\pm a^2)^3}} = \sqrt{x^2\pm a^2} \pm \frac{a^2}{\sqrt{x^2\pm a^2}}.$

150. $\int \frac{dx}{x\sqrt{(x^2+a^2)^3}} = \frac{1}{a^2\sqrt{x^2+a^2}} - \frac{1}{a^3}\log\frac{a+\sqrt{x^2+a^2}}{x}.$

151. $\int \frac{dx}{x\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\sec^{-1}\frac{a}{x}.$

152. $\int \frac{dx}{x^2\sqrt{(x^2\pm a^2)^3}} = -\frac{1}{a^4}\left[\frac{\sqrt{x^2\pm a^2}}{x} + \frac{x}{\sqrt{x^2\pm a^2}}\right].$

153. $\int \frac{dx}{x^3\sqrt{(x^2+a^2)^3}} = -\frac{1}{2a^2x^2\sqrt{x^2+a^2}} - \frac{3}{2a^4\sqrt{x^2+a^2}}$

$$+\frac{3}{2a^5}\log\frac{a+\sqrt{x^2+a^2}}{x}.$$

154. $\int \frac{dx}{x^3\sqrt{(x^2-a^2)^3}} = \frac{1}{2a^2x^2\sqrt{x^2-a^2}} - \frac{3}{2a^4\sqrt{x^2-a^2}}$

$$-\frac{3}{2a^5}\sec^{-1}\frac{a}{x}.$$

155. $\int f(x, \sqrt{x^2+a^2})\,dx = a\int f(a\tan u, a\sec u)\sec^2 u\,du$

$(x = a\tan u)$

156. $\int f(x, \sqrt{x^2-a^2})\,dx = a\int f(a\sec u, a\tan u)\sec u\tan u\,du$

$(x = a\sec u).$

Forms Containing $\sqrt{a^2-x^2}$

157. $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right].$

158. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \text{ or } -\cos^{-1}\left(\frac{x}{a}\right).$

159. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$

160. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} - a\log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right).$

161. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}.$

162. $\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}\sqrt{(a^2-x^2)^3}.$

163. $\int \sqrt{(a^2-x^2)^3}\,dx =$

$$\frac{1}{4}\left[x\sqrt{(a^2-x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2-x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{a}\right].$$

164. $\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}.$

165. $\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}}.$

166. $\int x\sqrt{(a^2-x^2)^3}\,dx = -\tfrac{1}{5}\sqrt{(a^2-x^2)^5}.$

167. $\int x^2\sqrt{a^2-x^2}\,dx =$

$$-\frac{x}{4}\sqrt{(a^2-x^2)^3}+\frac{a^2}{8}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right).$$

168. $\int x^3\sqrt{a^2-x^2}\,dx = (-\tfrac{1}{5}x^2-\tfrac{2}{15}a^2)\sqrt{(a^2-x^2)^3}.$

169. $\int x^2\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2-x^2)^5}$

$$+\frac{a^2x}{24}\sqrt{(a^2-x^2)^3}+\frac{a^4x}{16}\sqrt{a^2-x^2}+\frac{a^6}{16}\arcsin\frac{x}{a}.$$

170. $\int x^3\sqrt{(a^2-x^2)^3}\,dx = \frac{1}{7}\sqrt{(a^2-x^2)^7}-\frac{a^2}{5}\sqrt{(a^2-x^2)^5}.$

171. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

172. $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}.$

173. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}.$

173a. $\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2}+\frac{1}{2a}\log\frac{a+\sqrt{a^2-x^2}}{x}.$

174. $\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\frac{x}{a}.$

175. $\int \frac{x^3dx}{\sqrt{a^2-x^2}} = -\frac{2}{3}(a^2-x^2)^{\frac{1}{2}}-x^2(a^2-x^2)^{\frac{1}{2}}.$

176. $\int \frac{x^3dx}{\sqrt{(a^2-x^2)^3}} = 2(a^2-x^2)^{\frac{1}{2}}+\frac{x^2}{(a^2-x^2)^{\frac{1}{2}}}.$

177. $\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2}-\frac{1}{2a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}.$

177a. $\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}}+\frac{1}{a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}.$

178. $\int \frac{dx}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left[-\frac{\sqrt{a^2-x^2}}{x}+\frac{x}{\sqrt{a^2-x^2}}\right].$

179. $\int \frac{dx}{x^3\sqrt{(a^2-x^2)^3}} = -\frac{1}{2a^2x^2\sqrt{a^2-x^2}}+\frac{3}{2a^4\sqrt{a^2-x^2}}$

$$-\frac{3}{2a^5}\log\frac{a+\sqrt{a^2-x^2}}{x}.$$

180. $\int f(x,\sqrt{a^2-x^2})\,dx = a\int f(a\sin u, a\cos u)\cos u\,du$

$$(x = a\sin u).$$

Forms Containing $\sqrt{a+bx+cx^2}$

$X = a+bx+cx^2$, $q = 4ac-b^2$, and $k = \frac{4c}{q}$.

181. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\log\left(\sqrt{X}+x\sqrt{c}+\frac{b}{2\sqrt{c}}\right).$ if $c>0$

182. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{c}}\sinh^{-1}\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right).$ if $c>0$.

183. $\int \frac{dx}{\sqrt{X}} = \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right).$ if $c<0$.

184. $\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx+b)}{q\sqrt{X}}.$

185. $\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx+b)}{3q\sqrt{X}}\left(\frac{1}{X}+2k\right).$

186. $\int \frac{dx}{X^n\sqrt{X}} = \frac{2(2cx+b)\sqrt{X}}{(2n-1)qX^n}+\frac{2k(n-1)}{2n-1}\int\frac{dx}{X^{n-1}\sqrt{X}}.$

187. $\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c}+\frac{1}{2k}\int\frac{dx}{\sqrt{X}}.$

188. $\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X+\frac{3}{2k}\right)+\frac{3}{8k^2}\int\frac{dx}{\sqrt{X}}.$

189. $\int X^2\sqrt{X}\,dx =$

$$\frac{(2cx+b)\sqrt{X}}{12c}\left(X^2+\frac{5X}{4k}+\frac{15}{8k^2}\right)+\frac{5}{16k^3}\int\frac{dx}{\sqrt{X}}.$$

190. $\int X^n\sqrt{X}\,dx = \frac{(2cx+b)X^n\sqrt{X}}{4(n+1)c}+\frac{2n+1}{2(n+1)k}\int\frac{X^ndx}{\sqrt{X}}.$

191. $\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c}-\frac{b}{2c}\int\frac{dx}{\sqrt{X}}.$

192. $\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}.$

193. $\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n}-\frac{b}{2c}\int\frac{dx}{X^n\sqrt{X}}.$

194. $\displaystyle\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2\,c} - \frac{3\,b}{4\,c^2}\right)\sqrt{X} + \frac{3\,b^2 - 4\,ac}{8\,c^2}\int \frac{dx}{\sqrt{X}}.$

195. $\displaystyle\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2\,b^2 - 4\,ac)x + 2\,ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}.$

196. $\displaystyle\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2\,b^2 - 4\,ac)x + 2\,ab}{(2n-1)\,cq\,X^{n-1}\sqrt{X}} + \frac{4\,ac + (2\,n-3)b^2}{(2n-1)\,cq}\int \frac{dx}{X^{n-1}\sqrt{X}}.$

197. $\displaystyle\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3\,c} - \frac{5\,bx}{12\,c^2} + \frac{5\,b^2}{8\,c^3} - \frac{2\,a}{3\,c^2}\right)\sqrt{X} + \left(\frac{3\,ab}{4\,c^2} - \frac{5\,b^3}{16\,c^3}\right)\int \frac{dx}{\sqrt{X}}.$

198. $\displaystyle\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3\,c} - \frac{b}{2\,c}\int \sqrt{X}\,dx.$

199. $\displaystyle\int x\,X\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5\,c} - \frac{b}{2\,c}\int X\sqrt{X}\,dx.$

200. $\displaystyle\int \frac{x\,X^n\,dx}{\sqrt{X}} = \frac{X^n\sqrt{X}}{(2n+1)c} - \frac{b}{2\,c}\int \frac{X^n dx}{\sqrt{X}}.$

201. $\displaystyle\int x^2\sqrt{X}\,dx = \left(x - \frac{5\,b}{6\,c}\right)\frac{X\sqrt{X}}{4\,c} + \frac{5\,b^2 - 4\,ac}{16\,c^2}\int \sqrt{X}\,dx.$

202. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x} + \frac{b}{2\sqrt{a}}\right),$ if $a > 0$.

203. $\displaystyle\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx + 2\,a}{x\sqrt{b^2 - 4\,ac}}\right),$ if $a < 0$.

204. $\displaystyle\int \frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx},$ if $a = 0$.

205. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2\,a}\int \frac{dx}{x\sqrt{X}}.$

206. $\displaystyle\int \frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int \frac{dx}{\sqrt{X}} + a\int \frac{dx}{x\sqrt{X}}.$

207. $\displaystyle\int \frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + c\int \frac{dx}{\sqrt{X}}.$

Forms Involving $\sqrt{2ax - x^2}$

208. $\displaystyle\int \sqrt{2ax - x^2}\,dx =$

$$\tfrac{1}{2}[(x - a)\sqrt{2ax - x^2} + a^2\sin^{-1}(x - a)/a.].$$

209. $\displaystyle\int \frac{dx}{\sqrt{2\,ax - x^2}} = \cos^{-1}\left(\frac{a - x}{a}\right) = 2\sin^{-1}\sqrt{\frac{x}{2a}}.$

210. $\displaystyle\int x^n\sqrt{2ax - x^2}\,dx = -\frac{x^{n-1}(2ax - x^2)^{\frac{3}{2}}}{n+2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{2ax - x^2}\,dx \quad n \neq -2.$

211. $\displaystyle\int \frac{\sqrt{2ax - x^2}}{x^n}\,dx =$

$$\frac{(2ax - x^2)^{\frac{3}{2}}}{(3 - 2n)ax^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax - x^2}}{x^{n-1}}\,dx \quad n \neq \frac{3}{2}.$$

212. $\displaystyle\int \frac{x^n dx}{\sqrt{2ax - x^2}} =$

$$\frac{-\,x^{n-1}\sqrt{2ax - x^2}}{n} + \frac{a(2n-1)}{n}\int \frac{x^{n-1}}{\sqrt{2ax - x^2}}\,dx \quad n \neq 0.$$

213. $\displaystyle\int \frac{dx}{x^n\sqrt{2ax - x^2}} =$

$$\frac{\sqrt{2ax - x^2}}{a(1 - 2n)x^n} + \frac{n-1}{(2n-1)a}\int \frac{dx}{x^{n-1}\sqrt{2ax - x^2}} \quad n \neq \frac{1}{2}.$$

214. $\displaystyle\int \frac{dx}{(2ax - x^2)^{\frac{3}{2}}} = \frac{x - a}{a^2\sqrt{2ax - x^2}}.$

215. $\displaystyle\int \frac{x\,dx}{(2ax - x^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{2ax - x^2}}.$

216. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}} = \log(x + a + \sqrt{2ax + x^2}).$

Miscellaneous Algebraic Forms

217. $\displaystyle\int \sqrt{ax^2 + c}\,dx =$

$$\frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{a}}\log(x\sqrt{a} + \sqrt{ax^2 + c}), \quad [a > 0].$$

$$= \frac{x}{2}\sqrt{ax^2 + c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right) \quad [a < 0].$$

218. $\displaystyle\int \frac{dx}{\sqrt{a+bx}\cdot\sqrt{a'+b'x}} = \frac{2}{\sqrt{-bb'}}\tan^{-1}\sqrt{\frac{-b'(a+bx)}{b(a'+b'x)}}.$

219. $\displaystyle\int \sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1}x - \sqrt{1-x^2}.$

220. $\displaystyle\int \frac{dx}{\sqrt{a \pm 2bx + cx^2}} =$

$$\frac{1}{\sqrt{c}}\log\left(\pm b + cx + \sqrt{c}\,\sqrt{a \pm 2bx + cx^2}\right).$$

221. $\displaystyle\int \frac{dx}{\sqrt{a \pm 2bx - cx^2}} = \frac{1}{\sqrt{c}}\sin^{-1}\frac{cx \mp b}{\sqrt{b^2+ac}}.$

222. $\displaystyle\int \frac{x\,dx}{\sqrt{a \pm 2bx + cx^2}} = \frac{1}{c}\sqrt{a \pm 2bx + cx^2}$

$$\mp \frac{b}{\sqrt{c^3}}\log\left(\pm b + cx + \sqrt{c}\,\sqrt{a \pm 2bx + cx^2}\right).$$

223. $\displaystyle\int \frac{x\,dx}{\sqrt{a \pm 2bx - cx^2}} = -\frac{1}{c}\sqrt{a \pm 2bx - cx^2} \pm \frac{b}{\sqrt{c^3}}\sin^{-1}\frac{cx \mp b}{\sqrt{b^2+ac}}.$

Forms Involving Trigonometric Functions

224. $\displaystyle\int \sin x\,dx = -\cos x$, or versin x.

225. $\displaystyle\int \cos x\,dx = \sin x$, or $-$ coversin x.

226. $\displaystyle\int \tan x\,dx = -\log\cos x.$

227. $\displaystyle\int \cot x\,dx = \log\sin x.$

228. $\displaystyle\int \sec x\,dx = \log(\sec x + \tan x) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$

229. $\displaystyle\int \csc x\,dx = \log(\csc x - \cot x) = \log\tan\frac{x}{2}.$

230. $\displaystyle\int \sin^2 x\,dx = -\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}x = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x.$

231. $\displaystyle\int \sin^3 x\,dx = -\tfrac{1}{3}\cos x\,(\sin^2 x + 2).$

232. $\displaystyle\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

233. $\displaystyle\int \cos^2 x\,dx = \tfrac{1}{2}\sin x\cos x + \tfrac{1}{2}x = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x.$

234. $\displaystyle\int \cos^3 x\,dx = \tfrac{1}{3}\sin x\,(\cos^2 x + 2).$

235. $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$

236. $\displaystyle\int \sin\frac{x}{a}\,dx = -a\cos\frac{x}{a}.$

237. $\displaystyle\int \cos\frac{x}{a}\,dx = a\sin\frac{x}{a}.$

238. $\displaystyle\int \sin(a+bx)\,dx = -\frac{1}{b}\cos(a+bx).$

239. $\displaystyle\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx).$

240. $\displaystyle\int \frac{dx}{\sin x} = \int \csc x\,dx = \log(\csc x - \cot x)$

$$= -\frac{1}{2}\log\frac{1+\cos x}{1-\cos x} = \log\tan\frac{x}{2}.$$

241. $\displaystyle\int \frac{dx}{\cos x} = \int \sec x\,dx = \log(\sec x + \tan x)$

$$= \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right) = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

242. $\displaystyle\int \frac{dx}{\cos^2 x} = \int \sec^2 x\,dx = \tan x.$

243. $\displaystyle\int \frac{dx}{\cos^n x} = \frac{1}{n-1}\cdot\frac{\sin x}{\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}.$

244. $\displaystyle\int \frac{dx}{1 \pm \sin x} = \mp\tan\left(\frac{\pi}{4} \mp \frac{x}{2}\right).$

245. $\displaystyle\int \frac{dx}{1+\cos x} = \tan\frac{x}{2}.$

246. $\displaystyle\int \frac{dx}{1-\cos x} = -\cot\frac{x}{2}.$

247. $\displaystyle\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{a\tan\frac{1}{2}x + b}{\sqrt{a^2-b^2}}$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\frac{a\tan\frac{1}{2}x + b - \sqrt{b^2-a^2}}{a\tan\frac{1}{2}x + b + \sqrt{b^2-a^2}}.$$

248. $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a^2-b^2}\tan \frac{1}{2}x}{a+b}$$
$$= \frac{1}{\sqrt{b^2-a^2}} \log \left(\frac{\sqrt{b^2-a^2}\tan \frac{1}{2}x + a + b}{\sqrt{b^2-a^2}\tan \frac{1}{2}x - a - b} \right).$$

249. $$\int \frac{dx}{a+b\sin x + c\cos x}$$
$$= \begin{cases} \dfrac{1}{\sqrt{b^2+c^2-a^2}} \log \dfrac{b - \sqrt{b^2+c^2-a^2} + (a-c)\tan \frac{x}{2}}{b + \sqrt{b^2+c^2-a^2} + (a-c)\tan \frac{x}{2}} & \text{if } a^2 < b^2+c^2. \\ \dfrac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \dfrac{b + (a-c)\tan \frac{x}{2}}{\sqrt{a^2-b^2-c^2}} & \text{if } a^2 > b^2+c^2. \end{cases}$$

250. $$\int \sqrt{1-\cos x}\, dx = -2\sqrt{2}\cos \frac{x}{2}.$$

251. $$\int \sqrt{1+\cos x}\, dx = 2\sqrt{2}\sin \frac{x}{2}.$$

252. $$\int \sin mx \sin nx\, dx = \frac{\sin (m-n)x}{2(m-n)} - \frac{\sin (m+n)x}{2(m+n)}, \qquad [m^2 \neq n^2].$$

253. $$\int x\sin^2 x\, dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8}.$$

254. $$\int x^2 \sin^2 x\, dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x\cos 2x}{4}.$$

255. $$\int x\sin^3 x\, dx = \frac{x\cos 3x}{12} - \frac{\sin 3x}{36} - \frac{3}{4}x\cos x + \frac{3}{4}\sin x.$$

256. $$\int \sin^4 x\, dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}.$$

257. $$\int \cos mx \cos nx\, dx = \frac{\sin (m-n)x}{2(m-n)} + \frac{\sin (m+n)x}{2(m+n)}, \qquad [m^2 \neq n^2].$$

258. $$\int x\cos^2 x\, dx = \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}.$$

259. $$\int x^2\cos^2 x\, dx = \frac{x^3}{6} + \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x + \frac{x\cos 2x}{4}.$$

260. $$\int x\cos^3 x\, dx = \frac{x\sin 3x}{12} + \frac{\cos 3x}{36} + \frac{3}{4}x\sin x + \frac{3}{4}\cos x.$$

261. $$\int \cos^4 x\, dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32}.$$

262. $$\int \frac{\sin x\, dx}{x^m} = -\frac{\sin x}{(m-1)\,x^{m-1}} + \frac{1}{m-1}\int \frac{\cos x\, dx}{x^{m-1}}.$$

263. $$\int \frac{\cos x\, dx}{x^m} = -\frac{\cos x}{(m-1)x^{m-1}} - \frac{1}{m-1}\int \frac{\sin x\, dx}{x^{m-1}}.$$

264. $$\int \tan^3 x\, dx = \tfrac{1}{2}\tan^2 x + \log \cos x.$$

265. $$\int \tan^4 x\, dx = \tfrac{1}{3}\tan^3 x - \tan x + x.$$

265a. $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx, \qquad [n \neq 1].$$

266. $$\int \cot^3 x\, dx = -\tfrac{1}{2}\cot^2 x - \log \sin x.$$

267. $$\int \cot^4 x\, dx = -\tfrac{1}{3}\cot^3 x + \cot x + x.$$

268. $$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx, \qquad [n \neq 1].$$

269. $$\int \sin x\cos x\, dx = \tfrac{1}{2}\sin^2 x.$$

270. $$\int \sin mx \cos nx\, dx = -\frac{\cos (m-n)x}{2(m-n)} - \frac{\cos (m+n)x}{2(m+n)}.$$

271. $$\int \sin^2 x\cos^2 x\, dx = -\tfrac{1}{8}(\tfrac{1}{4}\sin 4x - x).$$

272. $$\int \sin x\cos^m x\, dx = -\frac{\cos^{m+1} x}{m+1}.$$

273. $$\int \sin^m x\cos x\, dx = \frac{\sin^{m+1} x}{m+1}.$$

274. $$\int \cos^m x \sin^n x\, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \cos^{m-2} x \sin^n x\, dx.$$

275. $\int \cos^m x \sin^n x\, dx =$

$$-\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \cos^m x \sin^{n-2} x\, dx.$$

276. $\int \frac{\cos^m x\, dx}{\sin^n x} =$

$$-\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1}\int \frac{\cos^m x\, dx}{\sin^{n-2} x}.$$

277. $\int \frac{\cos^m x\, dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n}\int \frac{\cos^{m-2} x\, dx}{\sin^n x}.$

278. $\int \frac{\sin^m x\, dx}{\cos^n x} = -\int \frac{\cos^m\left(\frac{\pi}{2}-x\right) d\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)}.$

279. $\int \frac{\sin x\, dx}{\cos^2 x} = \frac{1}{\cos x} = \sec x.$

280. $\int \frac{\sin^2 x\, dx}{\cos x} = -\sin x + \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

281. $\int \frac{\cos x\, dx}{\sin^2 x} = \frac{-1}{\sin x} = -\operatorname{cosec} x.$

282. $\int \frac{dx}{\sin x \cos x} = \log \tan x.$

283. $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \log \tan \frac{x}{2}.$

284. $\int \frac{dx}{\sin x \cos^n x} = \frac{1}{(n-1)\cos^{n-1} x} + \int \frac{dx}{\sin x \cos^{n-2} x},$

$[n \neq 1].$

285. $\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

286. $\int \frac{dx}{\sin^2 x \cos^2 x} = -2 \cot 2x.$

287. $\int \frac{dx}{\sin^m x \cos^n x} = -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1} x \cdot \cos^{n-1} x}$

$$+\frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x \cdot \cos^n x}.$$

288. $\int \frac{dx}{\sin^m x} = -\frac{1}{m-1}\cdot\frac{\cos x}{\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}.$

289. $\int \frac{dx}{\sin^2 x} = -\cot x.$

---

290. $\int \tan^2 x\, dx = \tan x - x.$

291. $\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.$

292. $\int \cot^2 x\, dx = -\cot x - x.$

293. $\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx.$

294. $\int \sec^2 x\, dx = \tan x.$

295. $\int \sec^n x\, dx = \int \frac{dx}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}.$

296. $\int \csc^2 x\, dx = -\cot x.$

297. $\int \csc^n x\, dx = \int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}.$

---

298. $\int x \sin x\, dx = \sin x - x \cos x.$

299. $\int x^2 \sin x\, dx = 2x \sin x - (x^2 - 2)\cos x.$

300. $\int x^3 \sin x\, dx = (3x^2 - 6)\sin x - (x^3 - 6x)\cos x.$

301. $\int x^m \sin x\, dx = -x^m \cos x + m\int x^{m-1} \cos x\, dx.$

302. $\int x \cos x\, dx = \cos x + x \sin x.$

303. $\int x^2 \cos x\, dx = 2x \cos x + (x^2 - 2)\sin x.$

304. $\int x^3 \cos x\, dx = (3x^2 - 6)\cos x + (x^3 - 6x)\sin x.$

305. $\int x^m \cos x\,dx = x^m \sin x - m \int x^{m-1} \sin x\,dx.$

306. $\int \frac{\sin x}{x}\,dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} \cdots$

307. $\int \frac{\cos x}{x}\,dx = \log x - \frac{x^2}{2 \cdot 2!} + \frac{x^4}{4 \cdot 4!} - \frac{x^6}{6 \cdot 6!} + \frac{x^8}{8 \cdot 8!} \cdots$

Forms Involving Inverse Trigonometric Functions

308. $\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

309. $\int \cos^{-1} x\,dx = x \cos^{-1} x - \sqrt{1 - x^2}.$

310. $\int \tan^{-1} x\,dx = x \tan^{-1} x - \tfrac{1}{2} \log (1 + x^2).$

311. $\int \cot^{-1} x\,dx = x \cot^{-1} x + \tfrac{1}{2} \log (1 + x^2).$

312. $\int \sec^{-1} x\,dx = x \sec^{-1} x - \log (x + \sqrt{x^2 - 1}).$

313. $\int \csc^{-1} x\,dx = x \csc^{-1} x + \log (x + \sqrt{x^2 - 1}).$

314. $\int \operatorname{vers}^{-1} x\,dx = (x - 1) \operatorname{vers}^{-1} x + \sqrt{2x - x^2}.$

315. $\int \sin^{-1} \frac{x}{a}\,dx = x \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2}.$

316. $\int \cos^{-1} \frac{x}{a}\,dx = x \cos^{-1} \frac{x}{a} - \sqrt{a^2 - x^2}.$

317. $\int \tan^{-1} \frac{x}{a}\,dx = x \tan^{-1} \frac{x}{a} - \frac{a}{2} \log (a^2 + x^2).$

318. $\int \cot^{-1} \frac{x}{a}\,dx = x \cot^{-1} \frac{x}{a} + \frac{a}{2} \log (a^2 + x^2).$

319. $\int (\sin^{-1} x)^2\,dx = x (\sin^{-1} x)^2 - 2x + 2\sqrt{1 - x^2}\,(\sin^{-1} x).$

320. $\int (\cos^{-1} x)^2\,dx = x (\cos^{-1} x)^2 - 2x - 2\sqrt{1 - x^2}\,(\cos^{-1} x).$

321. $\int x \cdot \sin^{-1} x\,dx = \tfrac{1}{4}[(2x^2 - 1) \sin^{-1} x + x\sqrt{1 - x^2}].$

321a. $\int x \cos^{-1} x\,dx = \tfrac{1}{4}[(2x^2 - 1) \cos^{-1} x - x\sqrt{1 - x^2}].$

322. $\int x^n \sin^{-1} x\,dx = \frac{x^{n+1} \sin^{-1} x}{n+1} - \frac{1}{n+1} \int \frac{x^{n+1}\,dx}{\sqrt{1 - x^2}}.$

323. $\int x^n \cos^{-1} x\,dx = \frac{x^{n+1} \cos^{-1} x}{n+1} + \frac{1}{n+1} \int \frac{x^{n+1}\,dx}{\sqrt{1 - x^2}}.$

324. $\int x^n \tan^{-1} x\,dx = \frac{x^{n+1} \tan^{-1} x}{n+1} - \frac{1}{n+1} \int \frac{x^{n+1}\,dx}{1 + x^2}.$

325. $\int \frac{\sin^{-1} x\,dx}{x^2} = \log \left(\frac{1 - \sqrt{1 - x^2}}{x}\right) - \frac{\sin^{-1} x}{x}.$

326. $\int \frac{\tan^{-1} x\,dx}{x^2} = \log x - \frac{1}{2} \log (1 + x^2) - \frac{\tan^{-1} x}{x}.$

Forms Involving Trigonometric Substitutions

327. $\int f(\sin x)\,dx = 2 \int f\left(\frac{2z}{1 + z^2}\right) \cdot \frac{dz}{1 + z^2}; \left(z = \tan \frac{x}{2}\right).$

328. $\int f(\cos x)\,dx = 2 \int f\left(\frac{1 - z^2}{1 + z^2}\right) \frac{dz}{1 + z^2}; \left(z = \tan \frac{x}{2}\right).$

329. $\int f(\sin x)\,dx = \int f(u) \frac{du}{\sqrt{1 - u^2}}; (u = \sin x).$

330. $\int f(\cos x)\,dx = -\int f(u) \frac{du}{\sqrt{1 - u^2}}; (u = \cos x).$

331. $\int f(\sin x, \cos x)\,dx = \int f(u, \sqrt{1 - u^2}) \frac{du}{\sqrt{1 - u^2}};$ $(u = \sin x).$

332. $\int f(\sin x, \cos x)\,dx = 2 \int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right) \frac{dz}{1+z^2}; \left(z = \tan \frac{x}{2}\right).$

333. $\int \frac{dx}{a + b \tan x} = \frac{1}{a^2 + b^2} [ax + b \log (a \cos x + b \sin x)].$

334. $\int \frac{dx}{a + b \cot x} = \frac{1}{a^2 + b^2} [ax - b \log (a \sin x + b \cos x)].$

Logarithmic Forms

335. $\int \log x\,dx = x \log x - x.$

336. $\int x \log x\,dx = \frac{x^2}{2} \log x - \frac{x^2}{4}.$

337. $\int x^2 \log x \, dx = \frac{x^3}{3} \log x - \frac{x^3}{9}.$

338. $\int x^p \log (ax) \, dx = \frac{x^{p+1}}{p+1} \log (ax) - \frac{x^{p+1}}{(p+1)^2} \qquad [p \neq -1].$

339. $\int (\log x)^2 \, dx = x (\log x)^2 - 2x \log x + 2x.$

340. $\int (\log x)^n \, dx = x (\log x)^n - n \int (\log x)^{n-1} \, dx, \quad [n \neq -1].$

341. $\int \frac{(\log x)^n}{x} \, dx = \frac{1}{n+1} (\log x)^{n+1}.$

342. $\int \frac{dx}{\log x} = \log (\log x) + \log x + \frac{(\log x)^2}{2 \cdot 2!} + \frac{(\log x)^3}{3 \cdot 3!} + \cdots$

343. $\int \frac{dx}{x \log x} = \log (\log x).$

344. $\int \frac{dx}{x (\log x)^n} = -\frac{1}{(n-1) (\log x)^{n-1}}.$

345. $\int \frac{x^m \, dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1) (\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m \, dx}{(\log x)^{n-1}}.$

346. $\int x^m \log x \, dx = x^{m+1} \left[\frac{\log x}{m+1} - \frac{1}{(m+1)^2} \right].$

347. $\int x^m (\log x)^n \, dx =$

$$\frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} \, dx, \quad [m, n \neq -1].$$

348. $\int \sin \log x \, dx = \tfrac{1}{2} x \sin \log x - \tfrac{1}{2} x \cos \log x.$

349. $\int \cos \log x \, dx = \tfrac{1}{2} x \sin \log x + \tfrac{1}{2} x \cos \log x.$

Exponential Forms

350. $\int e^x \, dx = e^x.$

351. $\int e^{-x} \, dx = -e^{-x}.$

352. $\int e^{ax} \, dx = \frac{e^{ax}}{a}.$

353. $\int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1).$

354. $\int x^m e^{ax} \, dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} \, dx.$

355. $\int \frac{e^{ax} \, dx}{x} = \log x + \frac{ax}{1!} + \frac{a^2x^2}{2 \cdot 2!} + \frac{a^3x^3}{3 \cdot 3!} + \cdots$

356. $\int \frac{e^{ax}}{x^m} \, dx = -\frac{1}{m-1} \frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1} \int \frac{e^{ax}}{x^{m-1}} \, dx.$

357. $\int e^{ax} \log x \, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} \, dx.$

358. $\int e^{ax} \cdot \sin px \, dx = \frac{e^{ax} (a \sin px - p \cos px)}{a^2 + p^2}.$

359. $\int e^{ax} \cdot \cos px \, dx = \frac{e^{ax} (a \cos px + p \sin px)}{a^2 + p^2}.$

360. $\int \frac{dx}{1 + e^x} = x - \log (1 + e^x) = \log \frac{e^x}{1 + e^x}.$

361. $\int \frac{dx}{a + be^{px}} = \frac{x}{a} - \frac{1}{ap} \log (a + be^{px}).$

362. $\int \frac{dx}{ae^{mx} + be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1} \left(e^{mx} \sqrt{\frac{a}{b}} \right).$

363. $\int e^{ax} \sin^n bx \, dx = \frac{1}{a^2 + n^2b^2} \Big((a \sin bx - nb \cos bx)$

$$e^{ax} \sin^{n-1} bx + n (n-1) b^2 \int e^{ax} \sin^{n-2} bx \cdot dx \Big).$$

364. $\int e^{ax} \cos^n bx \, dx = \frac{1}{a^2 + n^2b^2} \Big((a \cos bx + nb \sin bx)$

$$e^{ax} \cos^{n-1} bx + n (n-1) b^2 \int e^{ax} \cos^{n-2} bx \, dx \Big).$$

Hyperbolic Forms

365. $\int \sinh x \, dx = \cosh x.$

366. $\int \cosh x \, dx = \sinh x.$

367. $\int \tanh x \, dx = \log \cosh x.$

368. $\int \coth x \, dx = \log \sinh x.$

369. $\int \operatorname{sech} x\,dx = 2\tan^{-1}(e^x) = \tan^{-1}(\sinh x).$

370. $\int \operatorname{csch} x\,dx = \log\tanh\left(\frac{x}{2}\right).$

371. $\int x\sinh x\,dx = x\cosh x - \sinh x.$

372. $\int x\cosh x\,dx = x\sinh x - \cosh x.$

373. $\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x.$

374. $\int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x.$

375. $\int \sinh^2 x\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}.$

376. $\int \tanh^2 x\,dx = x - \tanh x.$

377. $\int \operatorname{sech}^2 x\,dx = \tanh x.$

378. $\int \cosh^2 x\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}.$

379. $\int \operatorname{ctnh}^2 x\,dx = x - \operatorname{ctnh} x.$

380. $\int \operatorname{csch}^2 x\,dx = -\operatorname{ctnh} x.$

381. $\int \sinh mx\sinh nx\,dx =$

$$\frac{\sinh(m+n)x}{2(m+n)} - \frac{\sinh(m-n)x}{2(m-n)} \qquad m^2 \neq n^2.$$

382. $\int \cosh mx\cosh nx\,dx =$

$$\frac{\sinh(m+n)x}{2(m+n)} + \frac{\sinh(m-n)x}{2(m-n)} \qquad m^2 \neq n^2.$$

383. $\int \sinh mx\cosh nx\,dx =$

$$\frac{\cosh(m+n)x}{2(m+n)} + \frac{\cosh(m-n)x}{2(m-n)} \qquad m^2 \neq n^2.$$

384. $\int \arg\sinh\frac{x}{a}\,dx = x\arg\sinh\frac{x}{a} - \sqrt{x^2+a^2}.$

385. $\int x\arg\sinh\frac{x}{a}\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\arg\sinh\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}.$

386. $\int \arg\cosh\frac{x}{a}\,dx = x\arg\cosh\frac{x}{a} - \sqrt{x^2-a^2}, \left[\arg\cosh\frac{x}{a} > 0\right]$

$$= x\arg\cosh\frac{x}{a} + \sqrt{x^2-a^2}, \left[\arg\cosh\frac{x}{a} < 0\right].$$

387. $\int \arg\tanh\frac{x}{a}\,dx = x\arg\tanh\frac{x}{a} + \frac{a}{2}\log(a^2-x^2).$

388. $\int x\arg\tanh\frac{x}{a}\,dx = \frac{x^2-a^2}{2}\arg\tanh\frac{x}{a} + \frac{ax}{2}.$

DEFINITE INTEGRALS

389. $\int_0^\infty x^{n-1}e^{-x}\,dx = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma(n).$ (Gamma function).

390. $\Gamma(n)$ is finite if $n > 0$, $\Gamma(n+1) = n\Gamma(n)$.

391. $\Gamma(n)\cdot\Gamma(1-n) = \frac{\pi}{\sin n\pi}.$

392. $\Gamma(n) = (n-1)!$ if $n =$ integer > 0.

393. $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$

394. $\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)}{2^n}\sqrt{\pi}$, where n is an integer and > 0. (see values of $\Gamma(n)$ at end of integral table)

395. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \mathrm{B}(m,n).$ (Beta function).

396. $\mathrm{B}(m,n) = \mathrm{B}(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where m and n are any positive real numbers.

397. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$

398. $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \qquad [m > 1].$

399. $\int_0^\infty \frac{dx}{(1+x)x^p} = \pi\csc p\pi, \qquad [p < 1].$

400. $\int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi,$ $[p < 1].$

401. $\int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi},$ $[0 < p < 1].$

402. $\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}},$ $[0 < m < n].$

403. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi.$

404. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a > 0$; 0, if $a = 0$; $-\frac{\pi}{2}$, if $a < 0$.

405. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (n)} \cdot \frac{\pi}{2}$, [$n$ an even integer],

$= \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot n}$, [$n$ an odd integer]

$= \frac{1}{2}\sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)},$ $[n > -1].$

406. $\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}$, if $m > 0$; 0, if $m = 0$; $-\frac{\pi}{2}$, if $m < 0$.

407. $\int_0^\infty \frac{\cos x\,dx}{x} = \infty.$

408. $\int_0^\infty \frac{\tan x\,dx}{x} = \frac{\pi}{2}.$

409. $\int_0^\pi \sin kx \cdot \sin mx\,dx = \int_0^\pi \cos kx \cdot \cos mx\,dx = 0,$

[$k \neq m$; m, k = integers].

410. $\int_0^\infty \frac{\sin x \cos mx\,dx}{x} = 0$, if $m < -1$ or $m > 1$,

$= \frac{\pi}{4}$, if $m = \pm 1$; $= \frac{\pi}{2}$, if $m^2 < 1$.

411. $\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}.$

412. $\int_0^\infty \frac{\sin^2 x\,dx}{x^2} = \frac{\pi}{2}.$

413. $\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m},$ $[m > 0].$

$= \frac{\pi}{2}e^{m},$ $[m < 0].$

414. $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

415. $\int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \sqrt{\frac{\pi}{2}}.$

416. $\int_0^{\pi/2} \frac{dx}{1+a\cos x} = \frac{\cos^{-1} a}{\sqrt{1-a^2}},$ $[a < 1].$

417. $\int_0^{2\pi} \frac{dx}{1+a\cos x} = \frac{2\pi}{\sqrt{1-a^2}},$ $(a^2 < 1].$

418. $\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log \frac{b}{a}.$

419. $\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab}.$

420. $\int_0^{\pi/2} \sin^{n-1} x \cos^{m-1} x\,dx = \frac{1}{2}\mathrm{B}\left(\frac{n}{2}, \frac{m}{2}\right),$

m and n positive integers.

421. $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}.$ $[a > 0].$

422. $\int_0^\infty x^n e^{-ax}\,dx = \frac{\Gamma(n+1)}{a^{n+1}},$ $[n > -1, a > 0],$

$= \frac{n!}{a^{n+1}},$ [n pos. integ., $a > 0$].

423. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right),$ $[a > 0]$

424. $\int_0^\infty xe^{-x^2}\,dx = \tfrac{1}{2}.$

425. $\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}.$

426. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$

427. $\displaystyle\int_0^\infty e^{(-x^2-a^2/x^2)}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}.$

428. $\displaystyle\int_0^\infty e^{-nx}\sqrt{x}\,dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}.$

429. $\displaystyle\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}.$

430. $\displaystyle\int_0^\infty e^{-ax}\cos mx\,dx = \frac{a}{a^2+m^2}, \qquad [a>0].$

431. $\displaystyle\int_0^\infty e^{-ax}\sin mx\,dx = \frac{m}{a^2+m^2}, \qquad [a>0].$

432. $\displaystyle\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}\cdot e^{-b^2/4a^2}}{2a}, \qquad [a>0].$

433. $\displaystyle\int_0^1 (\log x)^n\,dx = (-1)^n\cdot n!.$

434. $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}}dx = \frac{\sqrt{\pi}}{2}.$

435. $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^{-\frac{1}{2}}dx = \sqrt{\pi}.$

436. $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^n dx = n!.$

437. $\displaystyle\int_0^1 x\log(1-x)\,dx = -\tfrac{3}{4}.$

438. $\displaystyle\int_0^1 x\log(1+x)\,dx = \tfrac{1}{4}.$

439. $\displaystyle\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}.$

440. $\displaystyle\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}.$

441. $\displaystyle\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}.$

442. $\displaystyle\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}.$

443. $\displaystyle\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2.$

444. $\displaystyle\int_0^1 x^m\left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \text{ if } m+1>0,\ n+1>0.$

445. $\displaystyle\int_0^1 \frac{(x^p-x^q)\,dx}{\log x} = \log\left(\frac{p+1}{q+1}\right), \qquad [p+1>0,\ q+1>0].$

446. $\displaystyle\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}.$

447. $\displaystyle\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}.$

448. $\displaystyle\int_0^{\pi/2}\log\sin x\,dx = \int_0^{\pi/2}\log\cos x\,dx = -\frac{\pi}{2}\log 2.$

449. $\displaystyle\int_0^{\pi/2}\log\sec x\,dx = \int_0^{\pi/2}\log\csc x\,dx = \frac{\pi}{2}\log 2.$

450. $\displaystyle\int_0^{\pi} x\log\sin x\,dx = -\frac{\pi^2}{2}\log 2.$

451. $\displaystyle\int_0^{\pi/2}\sin x\log\sin x\,dx = \log 2 - 1.$

452. $\displaystyle\int_0^{\pi/2}\log\tan x\,dx = 0.$

453. $\displaystyle\int_0^{\pi}\log(a\pm b\cos x)\,dx = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right), \qquad [a\geqq b].$

454. $\displaystyle\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}.$

455. $\displaystyle\int_0^\infty \frac{x\,dx}{\sinh ax} = \frac{\pi^2}{4a^2}.$

456. $\displaystyle\int_0^\infty e^{-ax}\cosh bx\,dx = \frac{a}{a^2-b^2} \qquad a>0.$

457. $\displaystyle\int_0^\infty e^{-ax}\sinh bx\,dx = \frac{b}{a^2-b^2} \qquad a>0.$

458. $\displaystyle\int_{+\infty}^1 \frac{e^{-xu}}{u}\,du = \gamma + \log x - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} - \cdots,$ where $\displaystyle\gamma = \lim_{z\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{z}-\log z\right) = 0.5772157\cdots, 0<x<\infty.$

459. $$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots\right], \text{ if } k^2 < 1.$$

460. $$\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\, dx = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right], \text{ if } k^2 < 1.$$

461. $$\int_0^{\infty} e^{-x} \log x\, dx = -\gamma = -0.5772157 \cdots$$

462. $$\int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x}\right) e^{-x} dx = \gamma = 0.5772157 \cdots$$

[Euler's Constant].

463. $$\int_0^{\infty} \frac{1}{x}\left(\frac{1}{1 + x} - e^{-x}\right) dx = \gamma = 0.5772157 \cdots$$