CHAPTER 1: LIMITS AND THEIR PROPERTIES

1.1 A PREVIEW OF CALCULUS

- LEARNING OBJECTIVES
  - Understand what calculus is and how it compares with precalculus
  - Understand that the tangent line problem is basic to calculus
  - Understand that the area problem is also basic to calculus

- What is Calculus?
  - Mathematics of
    - Change
      - Velocities
      - Accelerations
    - Tangent Lines
    - Slopes
    - Areas
    - Volumes
    - Arc Lengths
    - Centroids
    - Curvatures
    - A Variety of Other Concepts
      - Model real-life situations
  - General Strategy
    - The reformulation of precalculus through the use of a limit process
    - Think of calculus as a "limit machine" that involves 3 stages
      - 1st Stage: Precalculus mathematics
        - Example:
          - The slope of a line
          - The area of a rectangle
      - 2nd Stage: Limit Process
      - 3rd Stage: A new calculus Formation
        - Example:
          - Derivative
          - Integral

- The Tangent Line Problem
  - What You are Given:
    - A function f
    - A point P on its graph

- What You are Asked to Find:
  - An equation of the tangent line to the graph at point P
    - With the exception of vertical lines, this is equivalent to finding the SLOPE of the tangent line at P.
      - You can **approximate** this slope by using the point of tangency and a second point on the curve. Such a line is called a **secant line**

- The Area Problem
  - What you are asked to find:
    - The area of a plane region that is bounded by the graph of functions

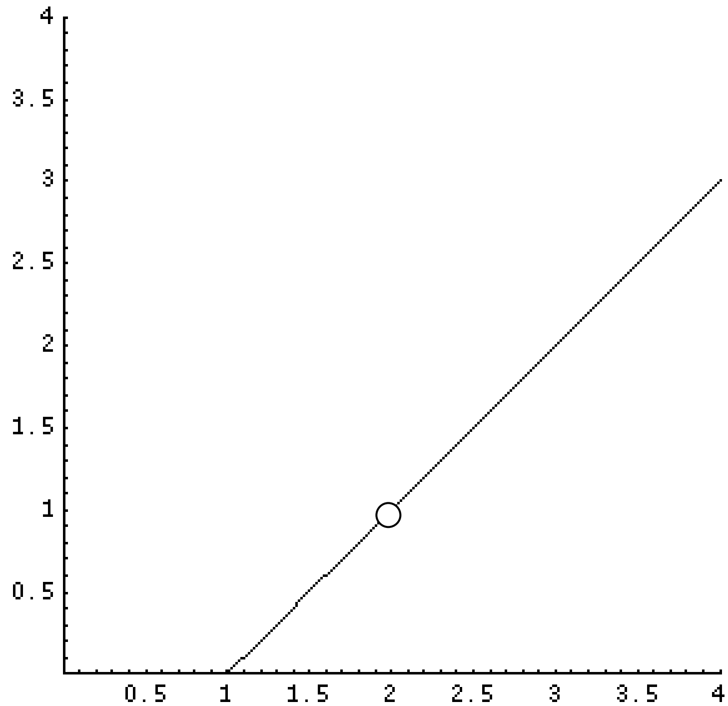
1.2 *FINDING LIMITS GRAPHICALLY AND NUMERICALLY*

- LEARNING OBJECTIVES
  - Estimate a limit using a numerical or graphic approach
  - Learn different ways that a limit can fail to exist
  - Study and use a formal definition of limit

- AN INTRODUCTION TO LIMITS
  - Consider the function $f(x)=\frac{x^2-3x+2}{x-2}$. The domain consists of all real numbers except for 2, or $\{x|x\in\Re \text{ and } x\neq 2\}$, or $(-\infty,2)\cup(2,\infty)$. When graphing the function, we could factor the top and cancel, keeping in mind there will be a break in the graph at $x=2$.
    So we have

$$
\begin{aligned}
f(x) &= \frac{x^2-3x+2}{x-2} \\
&= \frac{(x-2)(x-1)}{(x-2)} \\
&= x-1
\end{aligned}
$$

which is the graph of a line with an open circle at $x = 2$.
Let's first estimate the limit as x approaches 2 *graphically*. Or, writing this mathematically, $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$

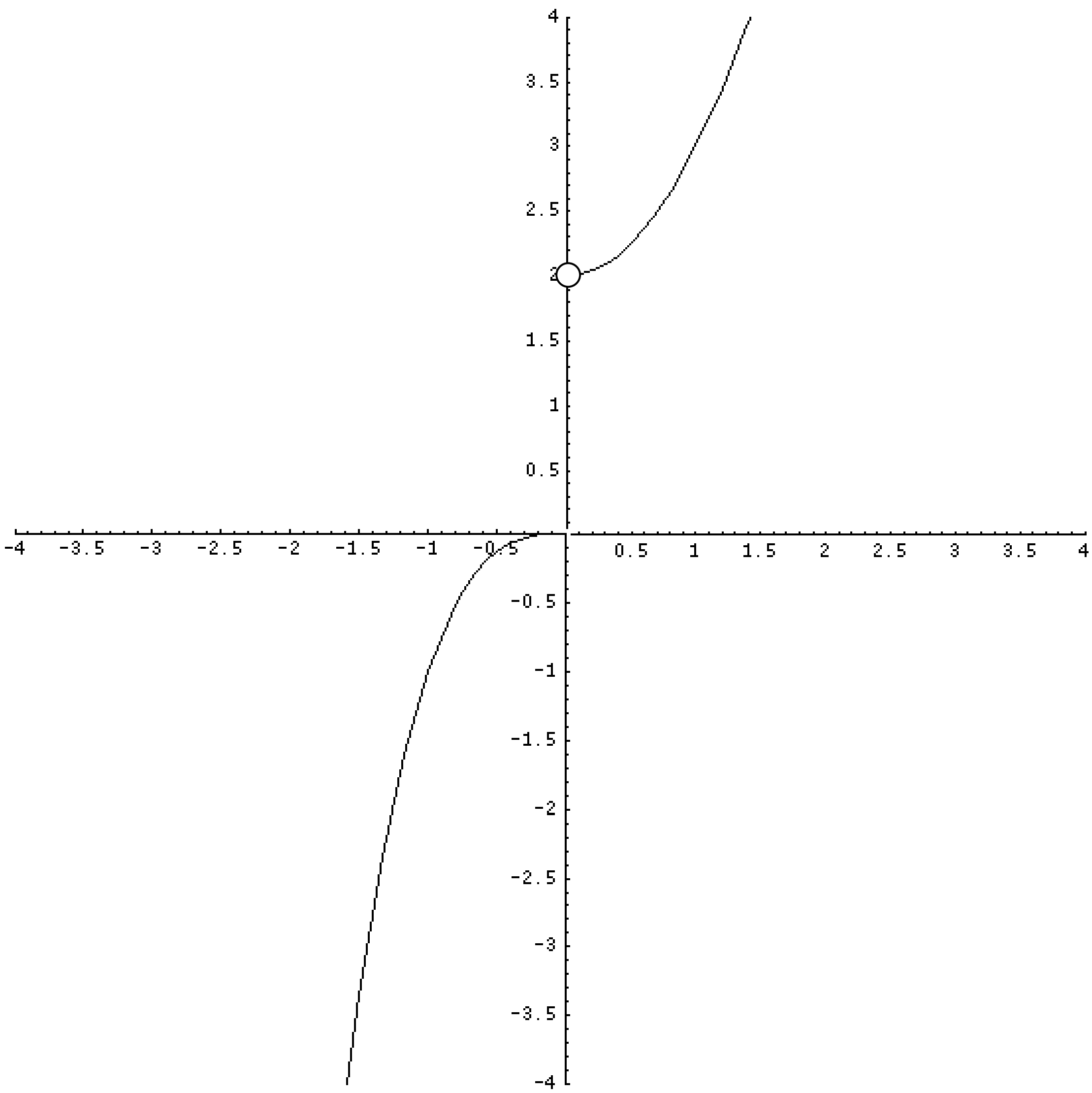


Now, let's estimate $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$ *numerically* using a table of values.

x	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25
$f(x) = \frac{x^2 - 3x + 2}{x - 2}$	0.75	0.9	0.99	0.999	?	1.001	1.01	1.1	1.25

Since the $f(x)$ is approaching 1 from the left and right of 2, we may conclude that $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = 1$.

- LIMITS THAT FAIL TO EXIST
    - Behavior that differs from the right to the left
        - Consider $\lim_{x \to 0} f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ x^3 & \text{if } x \le 0 \end{cases}$

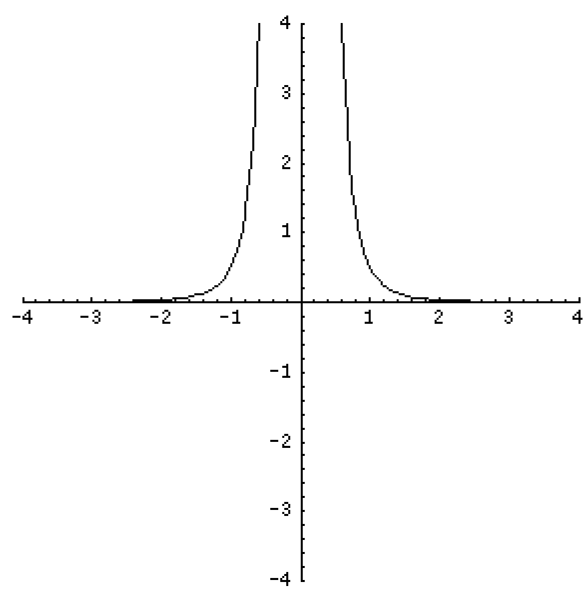


$\lim_{x \to 0^+} f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ x^3 & \text{if } x \le 0 \end{cases} = 2$ but $\lim_{x \to 0^-} f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ x^3 & \text{if } x \le 0 \end{cases} = 0$

- When there is a plus sign to the right of the limit that means you are approaching from the right, and when there is a minus sign to the right of the limit that means you are approaching from the left.
- Since the function approaches different values from the left and the right, $\lim_{x \to 0} f(x) = \begin{cases} x^2 + 2 & \text{if } x > 0 \\ x^3 & \text{if } x \le 0 \end{cases}$ does not exist.

  - Unbounded behavior

Consider the function $f(x) = \frac{1}{2x^4}$. This is a hyperbola with a vertical asymptote at $x = 0$.

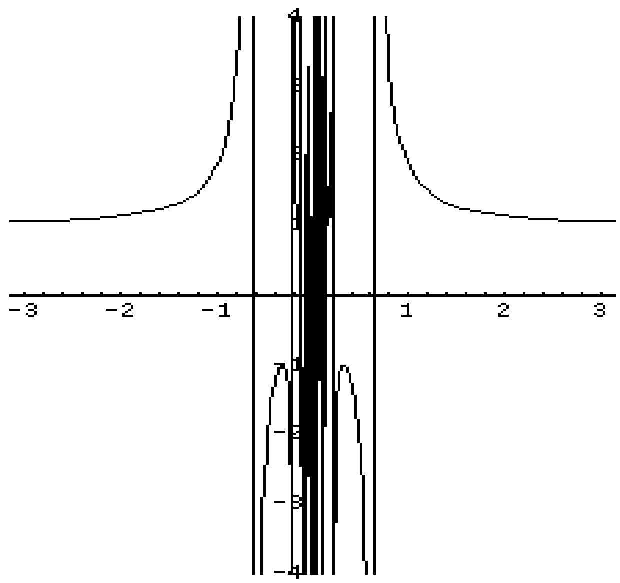


Notice that approaching from either the left or the right of 0 $f(x) = \frac{1}{2x^4}$ increases without bound, that is, $f(x)$ is approaching infinity, which is not an actual number. Therefore, we say

$$\lim_{x \to 0} \frac{1}{2x^4} \text{ does not exist, or DNE.}$$

- Oscillating behavior

  Consider the function $f(x) = \cos\frac{1}{x}$.



Let's examine what happens as x approaches 0.

x	$1/\pi$	$1/2\pi$	$1/3\pi$	$1/4\pi$	$1/5\pi$	$1/6\pi$	$1/7\pi$	$1/8\pi$	$1/9\pi$
$f(x) = \cos\frac{1}{x}$	-1	1	-1	1	-1	1	-1	1	-1

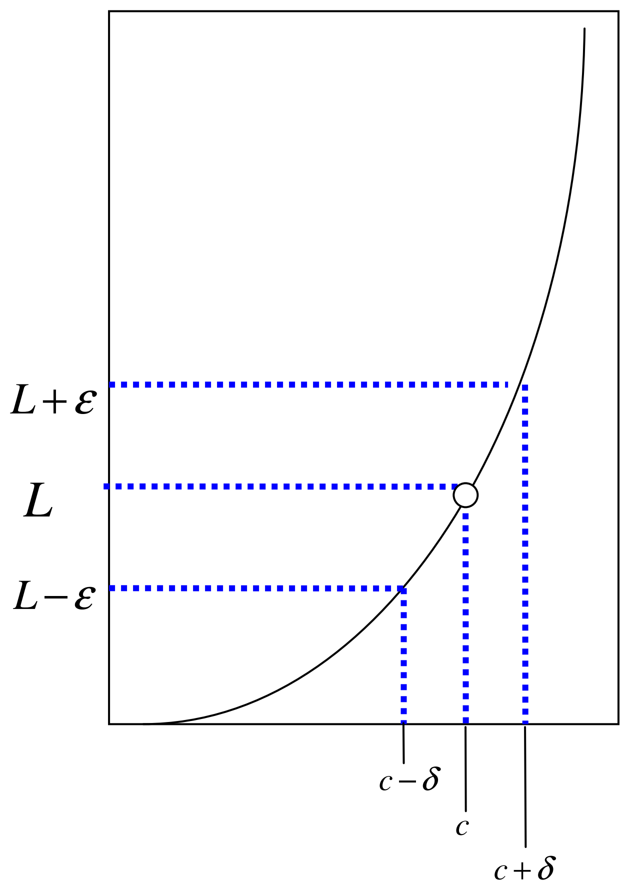
- Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of *c* than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches *c*.
3. $f(x)$ oscillates between two fixed values as x approaches *c*.

- A Formal Definition of Limit
  - Epsilon and delta
    - ε stands for epsilon and represents a small positive number. The phrase "$f(x)$ becomes arbitrarily close to L" means that $f(x)$ lies in the interval $(L-\varepsilon, L+\varepsilon)$ or using inequalities, $L-\varepsilon < f(x) < L+\varepsilon$

which is equal to $-\varepsilon < f(x) - L < \varepsilon$. This gives us the following absolute value: $|f(x) - L| < \varepsilon$

- δ stands for delta and represents a small positive number as well. The phrase "x becomes arbitrarily close to c" means that there exists a positive number δ such that x lies in either the interval $(c-\delta, c)$ or $(c, c+\delta)$. We can describe this using the double inequality $0 < |x - c| < \delta$. The first inequality $0 < |x - c|$ expresses the fact that $x \neq c$. The second inequality $|x - c| < \delta$ says that x is within a distance δ of c.
- The $\varepsilon - \delta$ definition



- DEFINITION OF LIMIT

Let f be a function defined on an open interval containing *c* (except possibly at *c*) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta,$ then $|f(x) - L| < \varepsilon$.

- Example: Finding a δ for a given ε
  - Given the limit $\lim_{x \to 4} \left(4 - \frac{x}{2}\right) = 2$, find δ such that $\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$ whenever $0 < |x - 4| < \delta$.
  - Solution: In this problem, we are working with a given value of ε. Specifically $\varepsilon = 0.01$. To find an appropriate δ, we simplify the inequality $\left|\left(4 - \frac{x}{2}\right) - 2\right| < 0.01$ to get $\left|\left(2 - \frac{x}{2}\right)\right| < 0.01$, or $\frac{1}{2}|(4 - x)| < 0.01$. This gives us $|(4 - x)| < 0.02$. Therefore, $\delta = 0.02$

- Example: Using the $\varepsilon - \delta$ definition of limit
  - Use the $\varepsilon - \delta$ definition of limit to prove that $\lim_{x \to 2} (3x - 2) = 4$.
  - Solution: Remember that the choice of δ depends on ε. We must show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|(3x - 2) - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$.
  - Now, $|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$. So for a given $\varepsilon > 0$ we can choose $\delta = \frac{\varepsilon}{3}$. This choice works because $0 < |x - 2| < \delta$ gives us $0 < |x - 2| < \frac{\varepsilon}{3}$ implies that $|(3x - 2) - 4| = 3|x - 2| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$

1.3 EVALUATING LIMITS ANALYTICALLY

- LEARNING OBJECTIVES
  - Evaluate a Limit Using Properties of Limits
  - Develop and Use a Strategy for Finding Limits
  - Evaluate a Limit Using Dividing Out and Rationalizing Techniques
  - Evaluate a Limit Using the Squeeze Theorem

- Properties of Limits
  - Some Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

  - Properties of Limits

Let b and c be real numbers, let n be a positive integer and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar Multiple: $\lim_{x \to c} \left[bf(x) \right] = bL$
2. Sum or Difference: $\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$
3. Product: $\lim_{x \to c} \left[f(x) g(x) \right] = LK$
4. Quotient: $\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} \left[f(x) \right]^n = L^n$

- Limits of Polynomial and Rational Functions

> - If p is a polynomial function and c is a real number, then $\lim_{x\to c} p(x) = p(c)$.
> - If r is a rational function given by $r(x) = \frac{p(x)}{q(x)}$ and c is a real number such that $q(c) \neq 0$, then $\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}$.

- The Limit of a Function Involving a Radical
  - Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

- The Limit of a Composite Function
  - If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(L)$.

- Limits of Trigonometric Functions

> Let c be a real number in the domain of the given trigonometric function.
>
> 1. $\lim_{x\to c} \sin x = \sin c$
> 2. $\lim_{x\to c} \cos x = \cos c$
> 3. $\lim_{x\to c} \tan x = \tan c$
> 4. $\lim_{x\to c} \cot x = \cot c$
> 5. $\lim_{x\to c} \sec x = \sec c$
> 6. $\lim_{x\to c} \csc x = \csc c$

- STRATEGIES FOR FINDING LIMITS

    - Functions That Agree at All But One Point

> Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$.

    - A Strategy for Finding Limits

> 1. Learn to recognize which limits can be evaluated by direct substitution.
> 2. If the limit as $f(x)$ as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$.
> 3. Apply $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c)$.
> 4. Use a graph or table to reinforce your conclusion.

- DIVIDING OUT AND RATIONALIZING TECHNIQUES

    - Dividing Out Techniques

$$\begin{aligned}\lim_{x\to 2}\frac{2-x}{x^2-4} &= \lim_{x\to 2}\frac{-(x-2)}{(x-2)(x+2)} \\ &= \lim_{x\to 2}\left(\frac{-1}{x+2}\right) \\ &= \lim_{x\to 2}\left(-\frac{1}{x+2}\right) \\ &= -\frac{1}{2+2} \\ &= -\frac{1}{4}\end{aligned}$$

- Rationalizing Techniques

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$$
$$=\lim_{x\to 0}\frac{(2+x)-(2)}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}}$$
$$=\frac{1}{\sqrt{2+0}+\sqrt{2}}$$
$$=\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{\sqrt{2}}{4}$$

- THE SQUEEZE THEOREM

If $h(x)\le f(x)\le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x\to c}h(x)=L=\lim_{x\to c}g(x)$, then $\lim_{x\to c}f(x)$ exists and is equal to L.

- TWO SPECIAL TRIGONOMETRIC LIMITS
  - $\lim_{x\to 0}\frac{\sin x}{x}=1$
  - $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

1.4 CONTINUITY AND ONE-SIDED LIMITS

- LEARNING OBJECTIVES
  - Determine continuity at a point and continuity on an open interval
  - Determine one-sided limits and continuity on a closed interval
  - Use properties of continuity
  - Understand and use the Intermediate Value Theorem

- CONTINUITY AT A POINT AND ON AN OPEN INTERVAL

  - Definition of Continuity

Continuity at a Point: A function f is **continuous at *c*** if the following **three** conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists.
3. $\lim_{x\to c} f(x) = f(c)$.

Continuity on an Open Interval: A function is **continuous on an open interval** (a,b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty,\infty)$ is **everywhere continuous**.

- ONE-SIDED LIMITS AND CONTINUITY A CLOSED INTERVAL
  - One-sided Limits
    - $\lim_{x\to c^+} f(x) = L$ means the limit as *x* approaches *c* from the right.
    - $\lim_{x\to c^-} f(x) = L$ means the limit as *x* approaches *c* from the left.
      - One-sided limits are useful in taking limits of functions involving radicals.
        - If *n* is an even integer, $\lim_{x\to 0^+} \sqrt[n]{x} = 0$
      - One-sided limits are useful in taking limits of step functions.
        - **Greatest Integer Function**, defined by

$$[\![x]\!] = \text{ greatest integer } n \text{ such that } n \le x.$$

- $[\![2.9]\!] = 2$

- $\lim_{x\to 0^-} [\![x]\!] = -1$ and $\lim_{x\to 0^+} [\![x]\!] = 0.$

- The Existence of a Limit

Let f be a function and let *c* and *L* be real numbers. The limit of $f(x)$ as *x* approaches *c* is *L* if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.

- Definition of Continuity on a Closed Interval

A function f is **continuous on a closed interval** $[a,b]$ if it is continuous on the open interval (a,b) and $\lim_{x\to c^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$. The function f is **continuous from the right** at *a* and **continuous from the left** at *b*.

- PROPERTIES OF CONTINUITY

If *b* is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at *c*.

1. Scalar multiple: bf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, if $g(c) \neq 0$

- EXAMPLES OF FUNCTIONS WHICH ARE CONTINUOUS AT EVERY POINT IN THEIR DOMAINS

1. Polynomial functions: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

2. Rational functions: $r(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$

3. Radical functions: $f(x) = \sqrt[n]{x}$

4. Trigonometric functions: $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x$

- CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at *c* and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at *c*.

- THE INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number *c* in $[a,b]$ such that $f(c) = k$.

1.5 *INFINITE LIMITS*

- LEARNING OBJECTIVES
    - Determine infinite limits from the left and from the right
    - Find and sketch the vertical asymptotes of the graph of a function

- DEFINITION OF INFINITE LIMITS

Let f be a function defined on an open interval containing *c* (except possibly at *c itself*). The statement $\lim_{x\to c} f(x) = \infty$ means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < \lvert x - c \rvert < \delta$. Similarly, the statement $\lim_{x\to c} f(x) = -\infty$ means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < \lvert x - c \rvert < \delta$. To define the **infinite limit from the left**, replace $0 < \lvert x - c \rvert < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right**, replace $0 < \lvert x - c \rvert < \delta$ by $c < x < c + \delta$.

- DEFINITION OF VERTICAL ASYMPTOTE

If $f(x)$ approaches infinity (or negative infinity) as *x* approaches *c* from the right or the left, then the line at $x = c$ is a **vertical asymptote** of the graph of f.

- A THEOREM CONCERNING VERTICAL ASYMPTOTES

> Let f and g be continuous on an open interval containing *c*. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing *c* such that $g(x) \neq 0$ for $x \neq c$ in the interval, then the graph of the function given by $h(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

- Properties of Infinite Limits

> Let c and L be real numbers and let f and g be functions such that
>
> $$\lim_{x \to c} f(x) = \infty \quad \text{and} \quad \lim_{x \to c} g(x) = L$$
>
> 1. Sum or Difference: $\lim_{x \to c} \left[f(x) \pm g(x) \right] = \infty$
> 2. Product: $\lim_{x \to c} \left[f(x) g(x) \right] = \infty, \; L > 0$
> 3. Quotient: $\lim_{x \to c} \left[\dfrac{g(x)}{f(x)} \right] = 0$
>
> Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as *x* approaches *c* is $-\infty$.