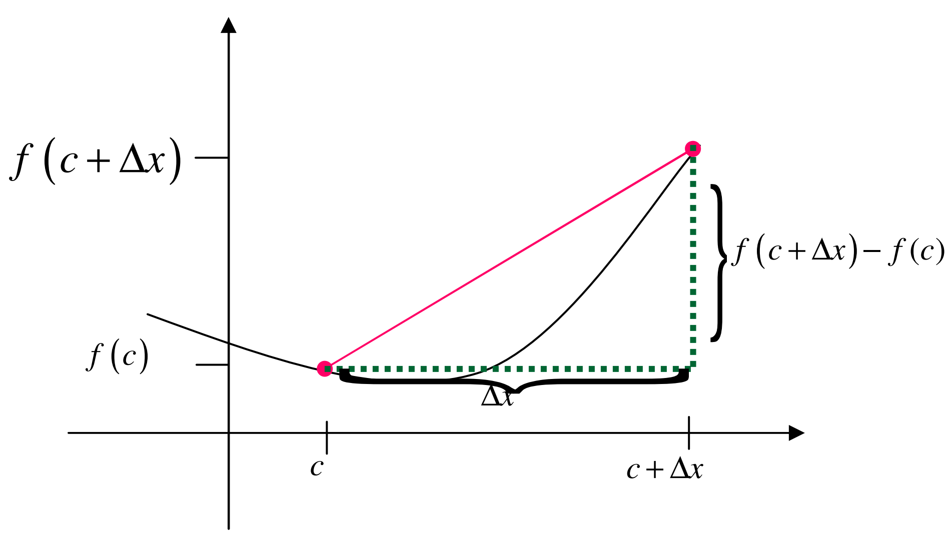
CHAPTER 2: DIFFERENTIATION

2.1 THE DERIVATIVE AND THE TANGENT LINE PROBLEM

- LEARNING OBJECTIVES
  - Find the slope of the tangent line to a curve at a point
  - Use the limit definition to find the derivative of a function
  - Understand the relationship between differentiability and continuity

- The Tangent Line Problem
  - Calculus grew out of four major problems that European mathematicians were working on during the 17^{th} century.
    - The tangent line problem (1.1 and 2.1)
      - Partial solutions were found be Descartes, Huygens, and Barrow
      - Credit for the first general solution is usually given to Isaac Newton and Gottfried Leibniz
    - The velocity and acceleration problem (2.2 and 2.3)
    - The minimum and maximum problem (3.1)
    - The area problem (1.1 and 4.2)
      - Each of these problems involves the notion of a limit

  - How do we find an equation of the tangent line to a graph at point P?
    - We can approximate this slope using a **secant line** through the point of tangency and a second point on the curve.



- Definition of Tangent Line with Slope *m*

> If f is defined on an open interval containing *c*, and if the limit $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = m$ exists, then the line passing through $(c, f(c))$ with slope *m* is the **tangent line** to the graph of f at the point $(c, f(c))$.

  - The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of** f **at** $x = c$.
    - Example: Find the slope of the graph of $f(x) = 3x + 5$ at the point $(-2, -1)$
    - Solution:

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{f(-2+\Delta x)-f(-2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\left[3(-2+\Delta x)+5\right]-\left[3(-2)+5\right]}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-6+3\Delta x+5-(-1)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-1+3\Delta x+1}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 3 \\ &= 3 \\ &= m\end{aligned}$$

- Derivative of a Function
  - Definition of the Derivative of a Function

> The **derivative** of f at x is given by $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ provided the limit exists. For all *x* for which this limit exists, f' is a function of *x*.

- Differentiability and Continuity
    - Alternative limit form of the derivative

      $$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

        - The existence of the limit in this alternative form requires that the one-sided limits $\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}$ and $\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$ exist and are equal. These one-sided limits are called the **derivatives from the left and from the right**, respectively. It follows that f is **differentiable on the closed interval** $[a,b]$ if it is differentiable on (a,b) and if the derivatives from the right at *a* and the derivative from the left at *b* both exist.

    - THEOREM: Differentiability Implies Continuity

If f is differentiable at $x=c$, then f is continuous at $x=c$.

2.2 BASIC DIFFERENTIATION RULES AND RATES OF CHANGE

- LEARNING OBJECTIVES

    - Find the derivative of a function using the Constant Rule
    - Find the derivative of a function using the Power Rule
    - Find the derivative of a function using the Constant Multiple Rule
    - Find the derivative of a function using the Sum and Difference Rules
    - Find the derivatives of the sine function and the cosine function
    - Use derivatives to find rates of change

- THE CONSTANT RULE

The derivative of a constant function is zero. That is, if *c* is a real number, then $\frac{d}{dx}(c)=0$

- Proof: Let $f(x)=c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}(c) &= f'(x)\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{c-c}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{0}{\Delta x}\\ &= \lim_{\Delta x\to 0} 0\\ &= 0\end{aligned}$$

  - Example: $s(t)=-9$

    $s'(t)=0$

- THE POWER RULE
  - If *n* is a rational number, then the function $f(x)=x^n$ is differentiable and $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$. For f to be differentiable at $x=0$, *n* must be a number such that x^{n-1} is defined on an interval containing 0.

    - Proof: Recall that the general binomial expansion for a positive integer *n* is $(x+\Delta x)^n = x^n + nx^{n-1}(\Delta x)+\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2+\cdots+(\Delta x)^n$. Let *n* be a positive integer greater than 1. The binomial expansion produces

$$\frac{d}{dx}\left(x^n\right) = \lim_{\Delta x \to 0} \frac{\left(x+\Delta x\right)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(x^n + nx^{n-1}\left(\Delta x\right) + \frac{n\left(n-1\right)x^{n-2}}{2}\left(\Delta x\right)^2 + \cdots + \left(\Delta x\right)^n\right) - \left(x^n\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(nx^{n-1} + \frac{n\left(n-1\right)x^{n-2}}{2}\left(\Delta x\right) + \cdots + \left(\Delta x\right)^{n-1}\right)$$

$$= nx^{n-1} + 0 + \cdots + 0$$

$$= nx^{n-1}$$

- Example: $f\left(x\right) = x^4$

  $$\frac{dy}{dx} = 4x^3$$

- THE CONSTANT MULTIPLE RULE

  - If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}\left[cf\left(x\right)\right] = cf'\left(x\right)$

  - Proof:

$$\frac{d}{dx}\left[cf\left(x\right)\right] = \lim_{\Delta x \to 0} \frac{cf\left(x+\Delta x\right) - cf\left(x\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} c\left(\frac{f\left(x+\Delta x\right) - f\left(x\right)}{\Delta x}\right)$$

$$= c\lim_{\Delta x \to 0}\left(\frac{f\left(x+\Delta x\right) - f\left(x\right)}{\Delta x}\right)$$

$$= cf'\left(x\right)$$

- Example: $y = -2x^{-5}$

  $y' = (-2)(-5)x^{-5-1}$

  $= 10x^{-6}$

- THE SUM AND DIFFERENCE RULES

  - The sum or difference of two differentiable functions f and g is itself differentiable. The derivative of the sum or difference of functions is the sum or difference of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right] = f'(x)-g'(x)$$

  - Proof:

$$\begin{aligned}\frac{d}{dx}\left[f(x)+g(x)\right] &= \lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)+g(x+\Delta x)\right]-\left[f(x)+g(x)\right]}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\ &= f'(x)+g'(x)\end{aligned}$$

- DERIVATIVES OF SINE AND COSINE FUNCTIONS

  - $\frac{\delta}{\delta x}(\sin x) = \cos x$ and $\frac{\delta}{\delta x}(\cos x) = -\sin x$

- Proof: Recall that $\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1$ and $\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$.

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sin x \cos \Delta x + \cos x \sin \Delta x) - (\sin x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x) \frac{\sin \Delta x}{\Delta x} - (\sin x) \frac{(1 - \cos \Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} (\cos x) \frac{\sin \Delta x}{\Delta x} - \lim_{\Delta x \to 0} (\sin x) \frac{(1 - \cos \Delta x)}{\Delta x} \\
&= (\cos x) \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} - (\sin x) \lim_{\Delta x \to 0} \frac{(1 - \cos \Delta x)}{\Delta x} \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x - 0 \\
&= \cos x
\end{aligned}$$

  - Example: $y = 4 \cos x$

    $$y' = (4)(-\sin x) = -4 \sin x$$

- RATES OF CHANGE
  - Velocity
    - Average Velocity
      - $\frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$
    - Instantaneous Velocity
      - $v'(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t)$

2.3 PRODUCT AND QUOTIENT RULES AND HIGHER-ORDER DERIVATIVES

- LEARNING OBJECTIVES
  - Find the derivative of a function using the Product Rule
  - Find the derivative of a function using the Quotient Rule
  - Find the derivative of a trigonometric function
  - Find a higher order derivative of a function

- THE PRODUCT RULE
  - The product of two differentiable functions f and g is itself differentiable. The derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}\left[f(x)g(x)\right]=f(x)g'(x)+g(x)f'(x)$$

- Proof:

$$\frac{d}{dx}\left[f(x)g(x)\right]=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)g(x+\Delta x)\right]-\left[f(x)g(x)\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)g(x+\Delta x)\right]-f(x+\Delta x)g(x)+f(x+\Delta x)g(x)-\left[f(x)g(x)\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)\left[g(x+\Delta x)-g(x)\right]+g(x)\left[f(x+\Delta x)-f(x)\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}+g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]+\lim_{\Delta x\to 0}\left[g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}g(x)\cdot\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=f(x)g'(x)+g(x)f'(x)$$

- Example: $y = -2x\sin x$

$$y' = -2x(\cos x) + (-2)\sin x$$
$$= -2(x\cos x + \sin x)$$

- Extension of the Product Rule

$$\frac{d}{dx}\left[f(x)g(x)h(x)\right] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

- Example: $y = x^3 \sin x \cos x$

$$\frac{dy}{dx} = (3x^2)\sin x\cos x + x^3(\cos x)\cos x + x^3\sin x(-\sin x)$$
$$= 3x^2\sin x\cos x + x^3\cos^2 x - x^3\sin^2 x$$
$$= 3x^2\sin x\cos x + x^3\left(\cos^2 x - \sin^2 x\right)$$

- THE QUOTIENT RULE
  - The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. The derivative of f/g is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2},\ g(x) \neq 0$$

Proof:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\lim_{\Delta x\to 0}\frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)}-\dfrac{f(x)}{g(x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{g(x)f(x+\Delta x)-f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{g(x)f(x+\Delta x)-f(x)g(x)+f(x)g(x)-f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\dfrac{g(x)\left[f(x+\Delta x)-f(x)\right]-f(x)\left[g(x+\Delta x)-g(x)\right]}{\Delta x}}{g(x)g(x+\Delta x)}$$

$$=\frac{\lim\limits_{\delta x\to 0}\dfrac{g(x)\left[f(x+\Delta x)-f(x)\right]}{\Delta x}-\lim\limits_{\Delta x\to 0}\dfrac{f(x)\left[g(x+\Delta x)-g(x)\right]}{\Delta x}}{\lim\limits_{\Delta x\to 0}g(x)g(x+\Delta x)}$$

$$=\frac{g(x)\lim\limits_{\delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}-f(x)\lim\limits_{\Delta x\to 0}\dfrac{g(x+\Delta x)-g(x)}{\Delta x}}{\lim\limits_{\Delta x\to 0}g(x)g(x+\Delta x)}$$

$$=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

- Example: $f(x)=\dfrac{2x-x^2}{x^3+2}$

$$f'(x) = \frac{(x^3+2)(2-2x)-(2x-x^2)(3x^2-0)}{(x^3+2)^2}$$

$$= \frac{(x^3+2)(2-2x)-(2x-x^2)(3x^2)}{(x^3+2)^2}$$

$$= \frac{2x^3-2x^4+4-4x-6x^3+3x^4}{(x^3+2)^2}$$

$$= \frac{x^4-4x^3-4x+4}{(x^3+2)^2}$$

- Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x$$
$$\frac{d}{dx}[\cot x] = -\csc^2 x$$
$$\frac{d}{dx}[\sec x] = \sec x \tan x$$
$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

- Higher Order Derivatives

○ First Derivative: y', $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}\left[f(x)\right]$, $D_x[y]$
○ Second Derivative: y'', $f''(x)$, $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}\left[f(x)\right]$, $D_x^{\,2}[y]$
○ Third Derivative: y''', $f'''(x)$, $\frac{d^3y}{dx^3}$, $\frac{d^3}{dx^3}\left[f(x)\right]$, $D_x^{\,3}[y]$
○ Fourth Derivative: $y^{(4)}$, $f^{(4)}(x)$, $\frac{d^4y}{dx^4}$, $\frac{d^4}{dx^4}\left[f(x)\right]$, $D_x^{\,4}[y]$
○ *n*th Derivative: $y^{(n)}$, $f^{(n)}(x)$, $\frac{d^ny}{dx^n}$, $\frac{d^n}{dx^n}\left[f(x)\right]$, $D_x^{\,n}[y]$

2.4 THE CHAIN RULE

- LEARNING OBJECTIVES
  - Find the derivative of a composite function using the Chain Rule
  - Find the derivative of a function using the General Power Rule
  - Simplify the derivative of a function using algebra
  - Find the derivative of a trigonometric function using the Chain Rule

- THE CHAIN RULE
  - If $y = f(u)$ is a differentiable function of *u* and $u = g(x)$ is a differentiable function of *x*, then $y = f(g(x))$ is a differentiable function of *x* and

    $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

    - Proof: Let $h(x) = f(g(x))$. Then using the alternative form of the derivative, you need to show that for $x = c,\ h'(c) = f'(g(c)) \cdot g'(c)$. We need to think about the behavior of g as *x* approaches *c*. We will assume that $g(x) \neq g(c)$ for values of *x* other than *c*. In Appendix A, the author shows how to use the differentiability of f and g to address the situation when there are other values of *x*, other than *c* such that $g(x) = g(c)$. Since g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$h'(c) = \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c}$$

$$= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right],\ g(x) \neq g(c)$$

$$= \left[\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right]$$

$$= f'(g(c))\, g'(c)$$

- THE CHAIN RULE
  - Example: $f(x) = (2x+5)^{10}$
  - Solution:

$$f'(x) = \left[10(2x+5)^{10-1} \right] \cdot (2+0)$$

$$= 20(2x+5)^{9}$$

- THE GENERAL POWER RULE

  - If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then $\frac{dy}{dx} = n[u(x)]^{n-1} \frac{du}{dx}$ or $\frac{d}{dx}[u^n] = nu^{n-1}u'$

  - Example: $y = (3x^3 + 5x)^6$
  - Solution:

$$y' = 6(3x^3 + 5x)^{6-1}(9x^2 + 5)$$

$$= 6(3x^3 + 5x)^5 (9x^2 + 5)$$

- DIFFERENTIATING FUNCTIONS INVOLVING RADICALS

  - Example: $s(t)=\sqrt[5]{\dfrac{1}{t^3-12}}$

  - Solution:

$$
\begin{aligned}
s(t)&=\sqrt[5]{\frac{1}{t^3-12}}\\
&=\sqrt[5]{\left(t^3-12\right)^{-1}}\\
&=\left(t^3-12\right)^{-1/5}\\
s'(t)&=-\frac{1}{5}\left(t^3-12\right)^{-1/5-1}\left(3t^2-0\right)\\
&=-\frac{1}{5}\left(t^3-12\right)^{-6/5}\left(3t^2\right)\\
&=-\frac{3}{5}t^2\left(t^3-12\right)^{-6/5}
\end{aligned}
$$

- TRIGONOMETRIC FUNCTIONS AND THE CHAIN RULE

  - The Chain Rule versions of the derivatives of the six trigonometric functions are as follows:

$\dfrac{d}{dx}[\sin u]=(\cos u)u'$	$\dfrac{d}{dx}[\cos u]=-(\sin u)u'$
$\dfrac{d}{dx}[\tan u]=\left(\sec^2 u\right)u'$	$\dfrac{d}{dx}[\cot u]=-\left(\csc^2 u\right)u'$
$\dfrac{d}{dx}[\sec u]=(\sec u\tan u)u'$	$\dfrac{d}{dx}[\csc u]=-(\csc u\cot u)u'$

- Example: $y = \sin 8x$
- Solution:

$$y = \sin \overbrace{8x}^{u}$$

$$y' = \underbrace{\cos \overbrace{8x}^{u}}_{\cos u}\overbrace{(8)}^{u'}$$

$$= 8\cos 8x$$

- Example: $y = \cot^3 x^2$
- Solution:

$$y = \cot^3 \overbrace{x^2}^{u}$$

$$= \left(\cot x^2\right)^3$$

$$\frac{d}{dx}\left[\left(\cot x^2\right)^3\right] = \underbrace{3\left(\cot x^2\right)^{3-1}}_{\text{derivative of outermost function}} \overbrace{\left(-\csc^2 x^2\right)}^{\text{derivative of trig function}} \quad \underbrace{(2x)}_{\text{derivative of the angle, which is what we called } u}$$

$$= -6x\left(\cot x^2\right)^2\left(\csc^2 x^2\right)$$

$$= -6x\cot^2 x^2 \csc^2 x^2$$

2.5 IMPLICIT DIFFERENTIATION

- LEARNING OBJECTIVES
  - Distinguish between functions written in implicit form and explicit form
  - Use implicit differentiation to find the derivative of a function

- IMPLICIT AND EXPLICIT FUNCTIONS
  - When you can write an equation with y on one side and a function of x on the other side, then you have an **explicit function**.
    - Example: $y = 3x^2 - 5x + 3$ is a function expressed in **explicit form**.

- Some functions are only implied by an equation.
  - Example: $x^2 - 2y^3 + 4y = 2$ is an **implicit equation** where it would be very difficult to express *y* as a function of *x* explicitly.

- DIFFERENTIATING WITH RESPECT TO x
  - So far, we have been solving equations for *y*, that is, we have been writing *y* **explicitly** as a function of *x*, and then differentiating.
    - Example: $2y = x^2 - 3$
    - Solution:

$$2y = x^2 - 3$$

$$y = \frac{x^2}{2} - \frac{3}{2}$$

$$\frac{d}{dx}[y] = \frac{d}{dx}\left[\frac{x^2}{2} - \frac{3}{2}\right] \quad \text{(we take the derivative of both sides)}$$

$$\frac{dy}{dx} = \frac{1}{2}(2x) - 0$$

$$\frac{dy}{dx} = x$$

    - Example: Find the derivative of $x^2 - 2y^3 + 4y = 2$ with respect to *x*

- Solution:

$$\frac{d}{dx}\left[x^2 - 2y^3 + 4y\right] = \frac{d}{dx}[2]$$

$$\frac{d}{dx}\left[x^2\right] - \frac{d}{dx}\left[2y^3\right] + \frac{d}{dx}[4y] = 0$$

$$2x - \underbrace{\overbrace{2}^{\text{constant multiple}} \overbrace{\left[3(y)^2\right]}^{\text{derivative of outer function}} \cdot \overbrace{\frac{dy}{dx}}^{\frac{d}{dx}[y]}}_{\text{chain rule}} + 4\frac{dy}{dx} = 0$$

$$2x - 6y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$-6y^2\frac{dy}{dx} + 4\frac{dy}{dx} = -2x \quad \left(\text{solve for } \frac{dy}{dx}\right)$$

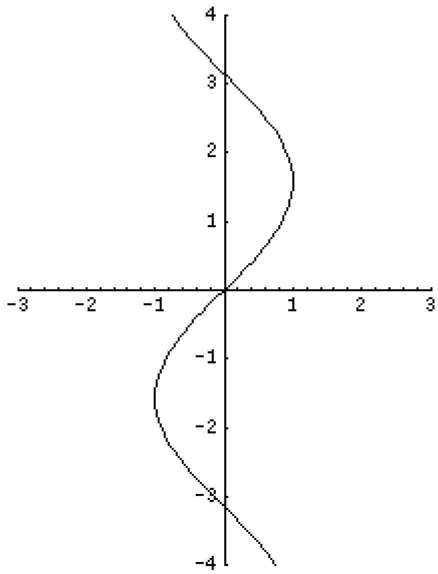
$$\frac{dy}{dx}\left(-6y^2 + 4\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{-6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{(-2)(x)}{(-2)\left(3y^2 - 2\right)}$$

$$\frac{dy}{dx} = \frac{x}{3y^2 - 2}$$

- DETERMINING A DIFFERENTIABLE FUNCTION
  - Example: Find $\frac{dy}{dx}$ implicitly for the equation $\sin y = x$. Then find the largest interval of the form $-a < y < a$ on which *y* is a differentiable function of *x*.



$$\sin y = x$$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$and$$

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

The largest interval about the origin for which *y* is a differentiable function of *x* is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

2.6 *RELATED RATES*

- LEARNING OBJECTIVES
  - Find a related rate
  - Use related rates to solve real-life problems

- FINDING RELATED RATES
  - We use the chain rule to **implicitly** find the rates of change of two or more **related variables** that are **changing** with respect to **time**.
  - Some common formulas used in this section

- Volume of a…
  - Sphere: $V = \frac{4}{3}\pi r^3$
  - Right Circular Cylinder: $V = \pi r^2 h$
  - Right Circular Cone: $V = \frac{1}{3}\pi r^2 h$
  - Rectangular Pyramid: $V = \frac{1}{3}lwh$
  - Pythagorean Theorem: $a^2 + b^2 = c^2$

- GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1.	Identify all **given** quantities and quantities **to be determined**. Make a sketch and label the quantities.
2.	Write an equation involving the variables whose rates of change either are given or are to be determined.
3.	Using the **Chain Rule**, implicitly differentiate both sides of the equation **with respect to time *t***.
4.	**After** completing step 3, substitute into the **resulting equation** all known values for the variables and their rates of change. Then solve for the required rate of change.

- Examples:
  - Vertical Motion. A ball is dropped from a height of 100 feet. One second later, another ball is dropped from a height of 75 feet. Which ball hits the ground first?

    Velocity: $s = -16t^2 + s_0$

    First ball:

$$0 = -16t^2 + 100$$

$$16t^2 = 100$$

$$t^2 = \frac{100}{16}$$

$$t = \sqrt{\frac{100}{16}}$$

$$t = \frac{10}{4}$$

$$t = 2.5 \text{ seconds to hit the ground}$$

Second ball:

$$0 = -16t^2 + 75$$

$$16t^2 = 75$$

$$t^2 = \frac{75}{16}$$

$$t = \sqrt{\frac{75}{16}}$$

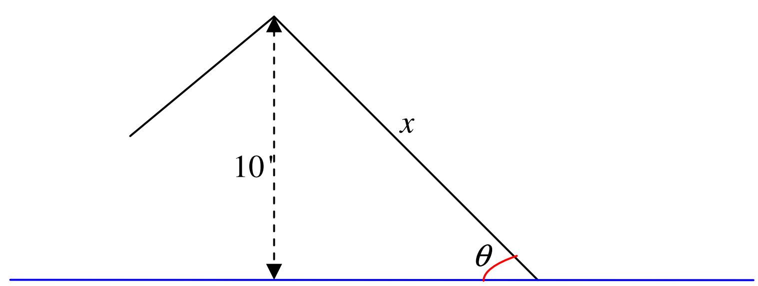
$$t = \frac{\sqrt{75}}{4}$$

$$t \approx 2.165 \text{ seconds to hit the ground}$$

$$t + 1 \approx 3.165 \text{ seconds to hit the ground}$$

Conclusion: The first ball will hit the ground first, and the second ball will hit the ground $3.165 - 2.5 = 0.665$ second later.

- Angle of Elevation. A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water. At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?



$$\sin\theta = \frac{10}{x} \quad (\text{we found})$$

$$\frac{dx}{dt} = (-1)\ ft/\sec \quad (\text{given})$$

$$\frac{d}{dt}[\sin\theta] = \frac{d}{dt}\left[\frac{10}{x}\right]$$

$$\cos\theta\frac{d\theta}{dt} = -\frac{10}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = -\frac{10}{x^2}\frac{dx}{dt}(\sec\theta) \quad \left(\frac{1}{\cos\theta} = \sec\theta\right)$$

$$\frac{d\theta}{dt} = -\frac{10}{25^2}(-1)\frac{25}{\sqrt{25^2-10^2}} \quad \left(\text{plugged in } \frac{dx}{dt} = -1\right), \ (\text{trig identities using triangle})$$

$$\frac{d\theta}{dt} = \frac{10}{25}\cdot\frac{1}{5\sqrt{21}}$$

$$\frac{d\theta}{dt} = \frac{2\sqrt{21}}{525}$$

$$\frac{d\theta}{dt} \approx 0.017 \text{ rad/sec}$$

Consider the following situation:

A container, in the shape of an inverted right circular cone, has a radius of 5 inches at the top and a height of 7 inches. At the instant when the water in the container is 6 inches deep, the surface level is falling at the rate of -1.3 in/s. Find the rate at which the water is being drained.