MATH 250/GRACEY NAME Key

WORKSHEET/4.2

- SIGMA NOTATION

The sum of n terms $a_1,\ldots,a_n$ is written as $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$ where i is the **index of summation**, a_i is the **ith term** of the sum, and the **upper and lower bounds of summation** are n and 1.

1. Find the sum.

$$\sum_{i=1}^{5} i^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2$$

$$= 1 + 4 + 9 + 16 + 25$$

$$= \boxed{55}$$

Summation Properties

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$ 2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Theorem: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$ 2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ 4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

2. Evaluate the following sums.

a. $\displaystyle\sum_{i=1}^{n}\left(3i-i^{2}\right)$ $\displaystyle = 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$

$$= 3\left(\frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{3n^2+3n}{2} - \frac{2n^3+3n^2+n}{6}$$

$$= \frac{9n^2+9n-2n^3-3n^2-n}{6} = \boxed{\frac{-2n^3+6n^2+8n}{6}}$$

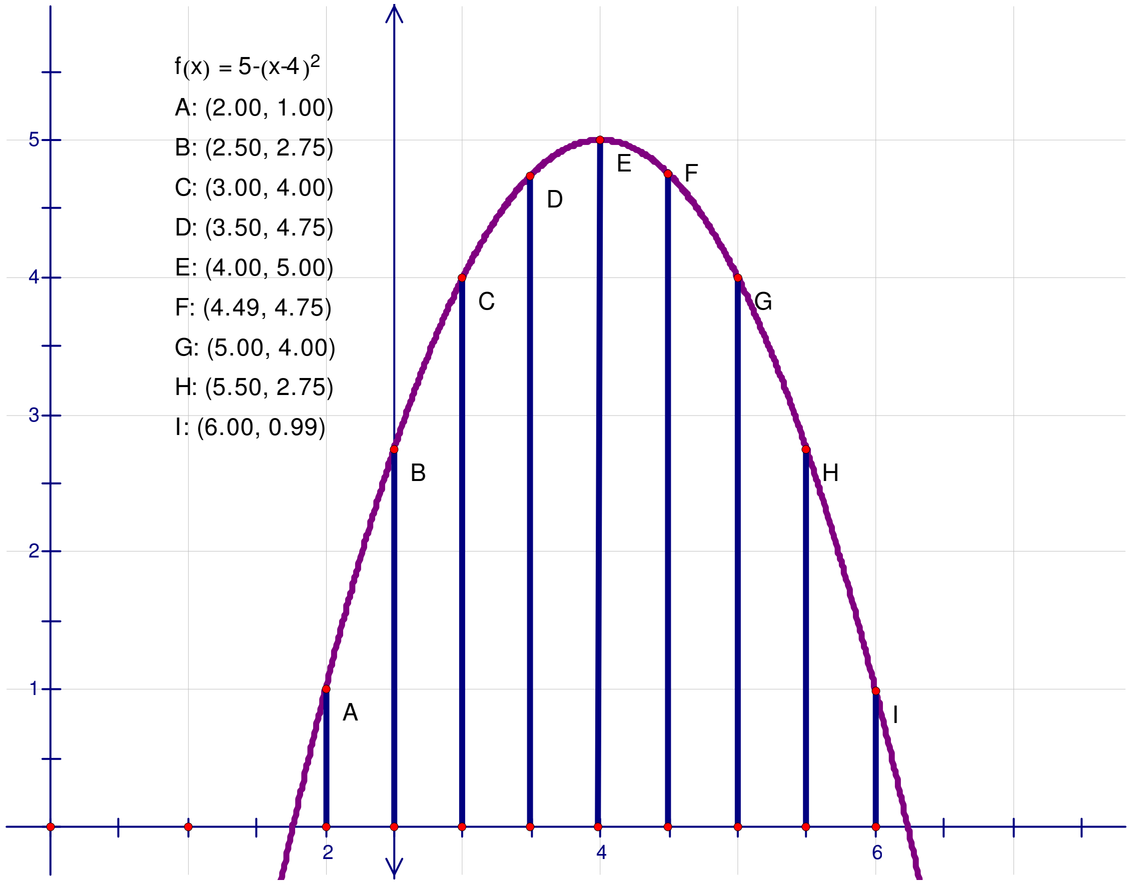
b. $\displaystyle\sum_{i=1}^{n}\left(6i+4i^{3}\right)$

Theorem: Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a,b]$. The limits as $n\to\infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x\\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = \dfrac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.



3. Find the upper and lower sums from $x = 2$ to $x = 6$.

f changes to decreasing

upper sum: width: $\frac{6-2}{8} = \frac{4}{8} = \frac{1}{2}$

$$= \frac{1}{2}\left[f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f(4) + f\left(\frac{9}{2}\right) + f(5) + f\left(\frac{11}{2}\right) \right]$$

$$= \frac{1}{2}\left(2.75 + 4 + 4.75 + 5 + 5 + 4.75 + 4 + 2.75 \right)$$

$$= \frac{1}{2}(35) = \boxed{16.5}$$

For the lower sum use the left endpoints while f is increasing and the right endpoints while f is decreasing

Definition of an Area in the Plane

Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the *x*-axis, and the vertical lines $x=a$ and $x=b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \qquad x_{i-1} \le c_i \le x_i$$

$$\text{where } \Delta x = \frac{b-a}{n},$$

right endpoint: $c_i = a + i\Delta x$, left endpoint: $c_i = a + (i-1)\Delta x$

4. Find the area of the region bounded by the graph $f(x) = x^3$, the *x*-axis, and the vertical lines $x=0$ and $x=1$.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$c_i = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^3}{n^3}\right)\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^4}_{\,n^2}} \left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty} \frac{1}{4n^2} \left(n^2 + 2n + 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n^2}^{\,1}}{4\cancel{n^2}} + \frac{2\cancel{n}}{4n^{\cancel{2}\,1}} + \frac{1}{4n^2}\right)$$

$$= \frac{1}{4} + 0 + 0 = \boxed{\frac{1}{4}}$$