MATH 251/GRACEY NAME______________________

WORKSHEET/8.5

PARTIAL FRACTION DECOMPOSITION

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (the degree of the numerator is greater than the degree of the denominator), divide the denominator into the numerator to obtain
$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$
Where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then

apply steps 2, 3, and 4 to the proper rational expression $\frac{N_1(x)}{D(x)}$.

2. **Factor the denominator:** Completely factor the denominator into factors of the form
$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$
Where ax^2+bx+c is irreducible.

3. **Linear factors:** For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of *m* fractions.
$$\frac{A_1}{(px+q)^1} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of *n* fractions.
$$\frac{B_1x+C_1}{(ax^2+bx+c)^1} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

1. Write the form of the partial fraction decomposition of the rational expression. Do not solve for the constants.

a. $$\frac{x}{(x^2+3)^3} = \frac{A_1x+B_1}{x^2+3} + \frac{A_2x+B_2}{(x^2+3)^2} + \frac{A_3x+B_3}{(x^2+3)^3}$$

b. $$\frac{3x^2-2}{x^4-4x^2+4} = \frac{A_1x+B_1}{x^2-2} + \frac{A_2x+B_2}{(x^2-2)^2}$$

$(x^2-2)^2$

Guidelines for solving the basic equation

1. Expand the basic equation.
2. Collect terms according to powers of *x*.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the resulting system of equations.

2. Rewrite the given rational expression as a sum of partial fractions.

$$\frac{4x^2+2x-1}{x^3+x^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} = \boxed{\frac{3}{x} - \frac{1}{x^2} + \frac{1}{x+1}}$$

$x^2(x+1)$

$$\frac{4x^2+2x-1}{x^2(x+1)} = \frac{A_1(x(x+1)) + A_2(x+1) + A_3(x^2)}{x^2(x+1)}$$

$$4x^2+2x-1 = A_1x^2 + A_1x + A_2x + A_2 + A_3x^2$$

$$4x^2+2x-1 = (A_1+A_3)x^2 + (A_1+A_2)x + A_2$$

$$\begin{cases} A_1 + A_3 = 4 \longrightarrow 3 + A_3 = 4 \longrightarrow A_3 = 1 \\ A_1 + A_2 = 2 \longrightarrow A_1 + (-1) = 2 \quad A_1 = 3 \\ A_2 = -1 \end{cases}$$

3. Find the integral.

a. $\int \frac{x+2}{x^2-4x}dx = \int \frac{x+2}{x(x-4)}dx = \int \left(-\frac{1}{2x} + \frac{3}{2(x-4)}\right)dx$

$= -\frac{1}{2}\ln|x| + \frac{3}{2}\ln|x-4| + C$

$= \boxed{\ln\left|\sqrt{\frac{(x-4)^3}{x}}\right| + C}$

$\frac{x+2}{x(x-4)} = \frac{A_1}{x} + \frac{A_2}{x-4} = -\frac{1}{2x} + \frac{3}{2(x-4)}$

$1x+2 = A_1x - 4A_1 + A_2x$

$1x+2 = (A_1+A_2)x - 4A_1$

$\begin{cases} A_1 + A_2 = 1 \rightarrow -\frac{1}{2} + A_2 = 1 \\ -4A_1 = 2 \end{cases}$

$A_2 = \frac{3}{2}$

$-4A_1 = 2$

$A_1 = -1/2$

b. $\int \frac{x^3-x+3}{x^2+x-2}dx = \int (x-1)dx + \int \frac{2x+1}{x^2+x-2}dx$

$= \frac{x^2}{2} - x + \int\left(\frac{1}{x+2} + \frac{1}{x-1}\right)dx$

$= \frac{x^2}{2} - x + \ln|x+2| + \ln|x-1| + C$

$= \frac{x^2}{2} - x + \ln|(x+2)(x-1)| + C$

$= \boxed{\frac{x^2}{2} - x + \ln|x^2+x-2| + C}$

$x - 1 + \frac{2x+1}{x^2+x-2}$

$(x^2+x-2)\overline{)x^3+0x^2-x+3}$

$-(x^3+x^2-2x)$

$-x^2+x+3$

$-(-x^2-x+2)$

$2x+1$

$\frac{2x+1}{x^2+x-2} = \frac{A_1}{x+2} + \frac{A_2}{x-1} = \frac{1}{x+2} + \frac{1}{x-1}$

$(x+2)(x-1)$

$2x+1 = A_1x - A_1 + A_2x + 2A_2$

$2x+1 = (A_1+A_2)x + (-A_1+2A_2)$

$\begin{cases} A_1 + A_2 = 2 \\ -A_1 + 2A_2 = 1 \end{cases} \rightarrow 3A_2 = 3 \rightarrow A_1 = 1$

$A_2 = 1$

c. $-\int \frac{-\sin x}{\cos x+\cos^2 x}\,dx = -\int \frac{du}{u+u^2} = -\int\left(\frac{1}{u}-\frac{1}{u+1}\right)du$

$u=\cos x$

$du = -\sin x\,dx$

$$\frac{1}{u(u+1)} = \frac{A_1}{u} + \frac{A_2}{u+1} = \frac{1}{u} - \frac{1}{u+1}$$

$0u+1 = A_1 u + A_1 + A_2 u$

$0u+1 = (A_1+A_2)u + A_1$

$A_1 + A_2 = 0 \rightarrow A_2 = -1$

$A_1 = 1$

$= -\left(\ln|u| - \ln|u+1|\right) + C$

$= -\ln|\cos x| + \ln|\cos x + 1| + C$

$= \ln\left|\frac{\cos x+1}{\cos x}\right| + C$

$= \boxed{\ln|1+\sec x| + C}$