Theorem: The Form of a Convergent Power Series

If *f* is represented by a power series $f(x)=\sum a_n (x-c)^n$ for all *x* in an open interval *I* containing *c*, then $a_n = \frac{f^{(n)}(c)}{n!}$ and

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Definitions of Taylor and Maclaurin Series

If a function *f* has derivatives of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

is called the **Taylor series for** $f(x)$ **at** ***c***. Moreover, if $c=0$, then the series is the **Maclaurin series for *f*.**

1. Use the definition to find the Taylor series centered at 1 for the function $f(x)=e^x$.

Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all *x* in the interval *I*, then the Taylor series for *f* converges to

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Prove that the Maclaurin series for $f(x)=\cosh x$ converges to the function for all *x*.

Guidelines for Finding a Taylor Series

1. Differentiate $f(x)$ several times and evaluate each derivative at *c*. Try to recognize a pattern in these numbers.
2. Use the sequence developed in step 1 to form the Taylor coefficients $a_n = \frac{f^{(n)}(c)}{n!}$ and determine the interval of convergence for the resulting power series $f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$
3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

Power Series for Elementary Functions

FUNCTION	INTERVAL OF CONVERGENCE
$\frac{1}{x}=1-(x-1)+(x-1)^2-(x-1)^3+\cdots+(-1)^n(x-1)^n+\cdots$	$0<x<2$
$\frac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots+(-1)^n(x-1)^n+\cdots$	$-1<x<1$
$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+\frac{(-1)^{n-1}(x-1)^n}{n}+\cdots$	$0<x\le 2$
$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$	$-\infty<x<\infty$
$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$	$-\infty<x<\infty$
$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$	$-\infty<x<\infty$
$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+\cdots$	$-1\le x\le 1$
$\arcsin x=x+\frac{x^3}{2\cdot 3}+\frac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\cdots+\frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)}+\cdots$	$-1\le x\le 1$

$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$ **(depending on the value of *k*)**

3. Find the Maclaurin series for the function $f(x) = x\cos x$.