Theorem 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Steps for using the Integral Test.

1. Conditions.
   a. Sketch a graph or give some rationale for why the function f is positive and continuous for $x \geq 1$.
   b. Find the derivative of f and figure out at which integer greater than one f begins to be a decreasing function.
2. Evaluate $\int_N^{\infty} f(x)\,dx$, where N is the integer for which f begins to be a decreasing function. Most often, N is one.
3. State your conclusion <u>in words</u> stating the reason for convergence or divergence.

1. Use the Integral Test to determine the convergence or divergence of the series.

   a. $\sum_{n=1}^{\infty} \frac{2}{3n+5}$

b. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

c. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Theorem 9.11 Convergence of *p*-Series

The *p*-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

1. Converges if $p > 1$, and
2. Diverges if $p \leq 1$.

2. Determine the convergence or divergence of the *p*-series.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt[5]{n^3}}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

c. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

3. Find the positive values of p for which the series converges.

a. $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$

b. $\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$