Math 251 Final Exam Review (Math Center) - Solutions

Chapter 7: Know the derivative and related integral formulas involving the inverse trigonometric functions, and be able to evaluate them by hand.

1. Find the equation of the line tangent to the curve $f(x) = \arctan\left(\frac{x}{2}\right)$ when x = $2\sqrt{3}$.

Slope: $f'(x) = \frac{1}{1+(x/2)^2} \cdot \frac{1}{2} = \frac{1}{2(1+x^2/4)} = \frac{1}{2+x^2/2} = \frac{2}{4+x^2} \Rightarrow$ $f'(2\sqrt{3}) = \frac{1}{8}$

Point: $(2\sqrt{3}, f(2\sqrt{3})) \Rightarrow \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3} \Rightarrow (2\sqrt{3}, \pi/3)$

$$\boxed{y - \pi/3 = \frac{1}{8}(x - 2\sqrt{3})}$$

2. Evaluate the integral: $\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$

$\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx \Rightarrow \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\arcsin u$

$\boxed{u = 3x}$

$du = 3dx$

$\boxed{\frac{1}{3}du = dx}$

$\Rightarrow \frac{1}{3}\arcsin(3x)\Big|_0^{1/6} = \frac{1}{3}\left[\arcsin \tfrac{1}{2} - \arcsin 0\right]$

$= \frac{1}{3}\left[\pi/6 - 0\right] = \boxed{\frac{\pi}{18}}$

3. Find the length of the curve $f(x) = \ln(\sin x)$ on the interval $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$. Integrate by hand.

$y = \ln(\sin x)$

$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$

$L = \int_{\pi/4}^{\pi/3} \sqrt{1 + \cot^2 x}\,dx$

$L = \int_{\pi/4}^{\pi/3} \sqrt{\csc^2 x}\,dx = \int_{\pi/4}^{\pi/3} \csc x\,dx$

$= \ln|\csc x - \cot x|\Big|_{\pi/4}^{\pi/3} = \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln|\sqrt{2} - 1|$

$= \boxed{\ln(1/\sqrt{3}) - \ln(\sqrt{2} - 1)}$ (other forms possible)

4. Find the area of the surface formed by revolving the graph of $f(x) = e^{2x}$ on the interval [0, 2] about the y – axis. You may use your calculator to integrate.

$f(x) = e^{2x}$

$f' = 2e^{2x}$

$(f')^2 = 4e^{4x}$

$$SA = 2\pi \int_0^2 x\sqrt{1+4e^{4x}}\,dx$$

$$= 2\pi(82.454) \approx \boxed{518.071}$$

5. A spring has a natural length of 6 inches. A force of 10 pounds compresses the spring 2 inches from it's natural length. Find the **work** done in stretching the spring from 7 inches to 10 inches.

$F = kx$

$10 = k(2)$

$k = 5$

$$W = \int_1^4 5x\,dx = \frac{5x^2}{2}\Big|_1^4$$

$$= 40 - 5/2 = \boxed{37.5 \text{ in-lbs.}}$$

.

6. $\int \frac{1}{4+9x^2}dx$

$u = 3x$

$du = 3\,dx$

$\frac{1}{3}du = dx$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{3}\arctan\frac{3x}{2} + C$$

$$= \boxed{\frac{1}{6}\arctan\frac{3x}{2} + C}$$

note: this problem can also be done w/ trigonometric substitution. The answer will look different...

7. $\int \frac{1}{4-9x^2}dx$

$$\int \frac{1}{4-9x^2}dx = \int\left[\frac{A}{2+3x} + \frac{B}{2-3x}\right]dx$$

$1 = A(2-3x) + B(2+3x)$

let $x = 2/3 \rightarrow 1 = 4B \rightarrow B = 1/4$

$x = -2/3 \rightarrow 1 = 4A \rightarrow A = 1/4$

$$\Rightarrow \frac{1}{4}\int\left[\frac{1}{2+3x} + \frac{1}{2-3x}\right]dx$$

$$= \frac{1}{4}\left[\frac{1}{3}\ln|2+3x| - \frac{1}{3}\ln|2-3x|\right]$$

$$\boxed{\frac{1}{12}\ln\left|\frac{2+3x}{2-3x}\right|}$$

8. $\int x^2 e^{2x} dx$ (int. by parts)

u	dv
x^2	e^{2x}
$2x$	$\frac{1}{2}e^{2x}$
2	$\frac{1}{4}e^{2x}$
0	$\frac{1}{8}e^{2x}$

$$\frac{x^2}{2}e^{2x} - \frac{2x}{4}e^{2x} + \frac{2}{8}e^{2x} + C$$

$$\boxed{\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C}$$

9. $\int \arcsin x\, dx$ (int. by parts)

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$

$$x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$\hookrightarrow u = 1-x^2$

$du = -2x\,dx$

$$x\arcsin x + \frac{1}{2}\int u^{-\frac{1}{2}} du$$

$$x\arcsin x + \frac{1}{2}\cdot 2u^{\frac{1}{2}} + C$$

$$\boxed{x\arcsin x + \sqrt{1-x^2} + C}$$

10. $\int \sqrt{1-x^2}\, dx$

$\sin\theta = x \to \cos\theta\, d\theta = dx$

$\cos\theta = \sqrt{1-x^2}$

$$\int \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{2}\left[\theta + \sin\theta\cos\theta\right] + C$$

$$\boxed{= \frac{1}{2}\left[\sin^{-1}x + x\sqrt{1-x^2}\right] + C}$$

11. $\int \sin^2(2x)\, dx$

$$= \int \frac{1-\cos 4x}{2} dx$$

$$= \frac{1}{2}\int (1-\cos 4x)\, dx$$

$$\boxed{= \frac{1}{2}\left[x - \frac{\sin 4x}{4}\right] + C}$$

12. $\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx$

$$\int_{-\pi}^{\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\,dx \Rightarrow \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{1}{3}\cdot 2u^{\frac{1}{2}}$$

$u = 4 + 3\sin x$

$du = 3\cos x\,dx$

$\frac{1}{3}du = \cos x\,dx$

$$\Rightarrow \frac{2}{3}\sqrt{4+3\sin x}\,\Big|_{-\pi}^{\pi}$$

$$= \frac{2}{3}\left[\sqrt{4} - \sqrt{4}\right] = \boxed{0}$$

*13. $\int_{1}^{2} \frac{x-2}{x-1}\,dx$ *Careful – this is an improper integral!

$$\int_{1}^{2} \frac{x-2}{x-1}\,dx = \lim_{b\to 1^+}\int_{b}^{2} \frac{x-2}{x-1}\,dx = \lim_{b\to 1^+}\int_{b}^{2}\left[1 - \frac{1}{x-1}\right]dx$$

$$\star \quad \begin{array}{r|l} & 1 \\ \hline x-1 & x-2 \\ & -(x-1) \\ \hline & -1 \end{array}$$

$$= \lim_{b\to 1^+}\Big[x - \ln|x-1|\Big]_{b}^{2}$$

$$= \lim_{b\to 1^+}\Big[(2-\ln(1)) - (2-\ln(b-1))\Big]$$

$$= -\infty \Rightarrow \boxed{\text{diverges}}$$

14. Find the particular solution, $y = f(x)$, to the differential equation $\frac{dy}{dx} = -x^2 y + y$, given $f(0) = 3$.

$\frac{dy}{dx} = -x^2y + y = y(-x^2+1) \Rightarrow \int \frac{dy}{y} = \int (-x^2+1)\,dx$

$\ln|y| = -x^3/3 + x + c \Rightarrow |y| = e^{-x^3/3 + x + c} \Rightarrow y = Ae^{-x^3/3 + x}$

$y(0) = 3 \Rightarrow 3 = Ae^0 \rightarrow A = 3 \Rightarrow \boxed{f(x) = 3e^{-x^3/3 + x}}$

15. A lake can support a maximum population of 2000 fish. The number of fish in the lake grows ***at a rate directly proportional to the difference between the maximum population and the current population.*** Initially (time $t = 0$ years) there are 50 fish in the lake. After 2 years, there are 80 fish.

a. Write and solve a differential equation to determine the population of fish in the lake at any time t.

P = population of fish

☆ $\frac{dP}{dt} = k(2000-P) \Rightarrow \int \frac{dP}{2000-P} = \int k\,dt \Rightarrow -\ln|2000-P| = kt + c,$

$\ln|2000-P| = -kt + c \Rightarrow |2000-P| = e^{(-kt+c)} \Rightarrow 2000 - P = Ae^{-kt}$

$P = 2000 - Ae^{-kt}$. Now, at time $t=0$, there are 50 fish.

$P = 2000 - Ae^{-kt}$, $P(0) = 50$

$50 = 2000 - Ae^0 \rightarrow A = 1950 \Rightarrow P = 2000 - 1950e^{-kt}$.

☆ at time $t = 2$, there are 80 fish.

$80 = 2000 - 1950e^{-k(2)} \rightarrow .9846 = e^{-2k} \rightarrow k = \frac{\ln .9846}{-2}$

$k \approx .00775$

$\boxed{☆ \text{ Eqn: } P(t) = 2000 - 1950e^{-.00775t}}$

b. How many fish will be in the lake at time $t = 5$ years?

$P(5) = 2000 - 1950e^{(-.00775 \cdot 5)} \approx 124.117$

$\boxed{\text{About } 124 \text{ fish}}$

16. Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{4\sqrt{n}}{n^2+3n+1}$

Use L.C.T. Compare to $\sum \frac{1}{n^{3/2}}$ (p-series, $p = 3/2 > 1 \Rightarrow$ converges)

$$\lim_{n\to\infty} \frac{\frac{4\sqrt{n}}{n^2+3n+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{4n^2}{n^2+3n+1} = 4 \quad \text{(finite, +)}$$

$$\boxed{\therefore \sum \frac{4\sqrt{n}}{n^2+3n+1} \ \underline{\text{converges}}}$$

17. Determine the interval of convergence: $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n \cdot 4^n}$

$\Rightarrow$ ☆ Root Test: $\lim_{n\to\infty} \sqrt[n]{\left| \frac{(x-3)^n}{n \cdot 4^n} \right|} = \lim_{n\to\infty} \frac{|x-3|}{\sqrt[n]{n} \cdot 4} = \frac{|x-3|}{4} < 1$

$\Rightarrow |x-3| < 4 \rightarrow -4 < x-3 < 4 \rightarrow -1 < x < 7$

Test endpts:

$x = -1$	$x = 7$
$\sum \frac{(-1)^n (-4)^n}{n \cdot 4^n}$	$\sum \frac{(-1)^n (4)^n}{n \cdot 4^n}$
$= \sum \frac{1}{n}$	$= \sum \frac{(-1)^n}{n}$
diverges	converges

$$\Rightarrow \boxed{\text{Interval: } (-1, 7]}$$

18. Derive the 3rd degree Taylor polynomial for $f(x) = \sqrt[3]{x}$ centered at $c = 8$.

	$c = 8$
$f(x) = x^{1/3}$	2
$f' = \frac{1}{3}x^{-2/3}$	$\frac{1}{12}$
$f'' = -\frac{2}{9}x^{-5/3}$	$-\frac{1}{144}$
$f''' = \frac{10}{27}x^{-8/3}$	$\frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{\frac{1}{144}(x-8)^2}{2!} + \frac{\frac{5}{3456}(x-8)^3}{3!}$$

$$\boxed{P_3(x) = 2 + \frac{(x-8)}{12} - \frac{(x-8)^2}{288} + \frac{5(x-8)^3}{20736}}$$

19. Use infinite series to evaluate the integral: $\int_0^{0.6} \sin(x^2)dx$ with an error of no more than 10^{-4}.

Clearly show all steps, and explain how you determined the number of terms necessary for this approximation. Give your final approximation correct to 5 decimal places.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

$\int_0^{0.6} \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\right) dx$

$= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots \Big|_0^{0.6}$

$= \frac{0.6^3}{3} - \frac{0.6^7}{7 \cdot 3!} + \frac{0.6^{11}}{11 \cdot 5!} - \frac{0.6^{15}}{15 \cdot 7!} + \dots$

ASRET: $\frac{.6^{11}}{11 \cdot 5!} < 10^{-4}$

$\therefore \int_0^{0.6} \sin(x^2)dx \approx \frac{0.6^3}{3} - \frac{0.6^7}{7 \cdot 3!} = \boxed{.07133}$

Chapter 12: Know how to work with equations in parametric and polar mode, including finding derivatives, arc length, and area.

20. Write the parametric equations for a line that passes through the points (4, -1) and (3, 5).

$\Delta x = 3 - 4 = -1$

$\Delta y = 5 - -1 = 6$

$$\boxed{\begin{aligned} x(t) &= -t + 4 \\ y(t) &= 6t - 1 \end{aligned}}$$

21. Convert the parametric equations to Cartesian, and identify the conic section represented.

$x(t) = 4 + \cos t,\ y(t) = 2\sin t$

$\cos t = x - 4$

$\sin t = y/2$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{y}{2}\right)^2 + (x-4)^2 = 1$

$$\boxed{\frac{y^2}{4} + \frac{(x-4)^2}{1} = 1 \quad \text{(ellipse)}}$$

22. Find the length of the curve represented by $x = \cos t - \sin t$, $y = \cos t + \sin t$, $0 \le t \le \pi$. Integrate by hand.

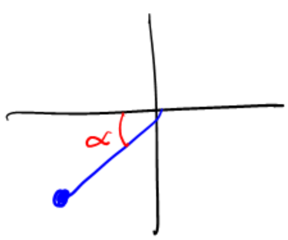
$$\boxed{\begin{aligned} \frac{dx}{dt} &= -\sin t - \cos t \\ \frac{dy}{dt} &= -\sin t + \cos t \end{aligned}}$$

$$\mathcal{L} = \int_0^\pi \sqrt{(-\sin t - \cos t)^2 + (-\sin t + \cos t)^2}\, dt$$

$$= \int_0^\pi \sqrt{\sin^2 t + 2\sin t\cos t + \cos^2 t + \sin^2 t - 2\sin t\cos t + \cos^2 t}\, dt$$

$$= \int_0^\pi \sqrt{2}\, dt = \boxed{\pi\sqrt{2}}$$

23. Express the Cartesian point $(-1, -\sqrt{3})$ in Polar Coordinates in two different ways. Include a plot of the point in your work.



$$r = \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\alpha = \sqrt{3}$$

$$\alpha = \pi/3 \rightarrow \theta = 4\pi/3$$

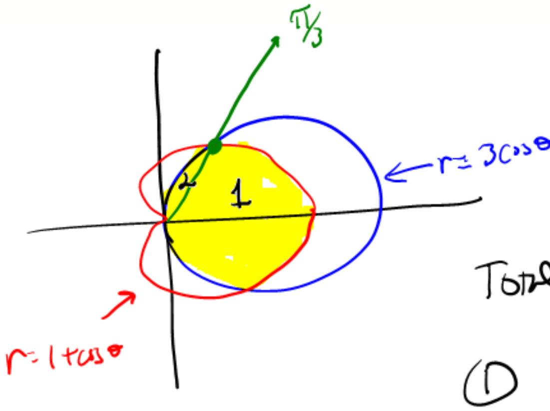
$$\boxed{\text{Polar: } (2; 4\pi/3)}$$

24. Convert the polar equation $r = 5\csc\theta$ to a Cartesian equation, and identify the shape (line, parabola, ellipse, circle, or hyperbola) represented.

$$r = \frac{5}{\sin\theta} \rightarrow r\sin\theta = 5$$

$$\boxed{y = 5} \qquad \text{(line)}$$

25. Find the area of the region that lies within both curves: $r = 1 + \cos\theta$, $r = 3\cos\theta$. You may use your calculator to evaluate your integrals, and give your final answer correct to 3 decimal places.



$1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta \rightarrow \cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}$

Total Area $= 2(① + ②)$

① $A = \frac{1}{2}\int_0^{\pi/3} (1+\cos\theta)^2 \, d\theta = 1.759677$

② $A = \frac{1}{2}\int_{\pi/3}^{\pi/2} (3\cos\theta)^2 \, d\theta = .2038167$

$A = 2(1.759677 + .2038167) \approx \boxed{3.927}$