CALCULUS III ✠ PRACTICE FINAL EXAM ✠ NAME: ______________________
TOTAL PAGES: 13

Instructions: ***Please show all work and circle your answers.***

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1. **(a)** The vector **v** has magnitude 8 and direction $q = 120°$. Find its component form.

(b) Suppose this same vector has as its initial point: $\left(-2, 7\sqrt{3}\right)$. Use your answer from part **(a)** to find its terminal point.

2. Determine if the following pairs of vector are orthogonal, parallel, or neither. Show your work.

(a) $\mathbf{v} = \langle 3, -2\rangle$ and $\mathbf{w} = \langle -1, 2\rangle$ **(b)** $\mathbf{v} = \langle -2, 0\rangle$ and $\mathbf{w} = \langle 0, 5\rangle$

(c) $\mathbf{v} = \langle -1, 2\rangle$ and $\mathbf{w} = \left\langle 0, -\frac{1}{2}\right\rangle$ **(d)** $\mathbf{v} = \langle 2, -3\rangle$ and $\mathbf{w} = \langle -2, 3\rangle$

3. Given the vectors $\mathbf{u} = \langle 2, -1, 1\rangle$ and $\mathbf{w} = \langle -3, 2, 2\rangle$…

(a) Calculate the angle (in degrees) between the vectors. Round answer to nearest hundredth.

(b) Find $\text{proj}_{\mathbf{u}}\mathbf{w}$.

4. ***Orthogonal vectors:***

(a) Find a vector orthogonal to the *yz*-plane.

(b) Find a vector orthogonal to the two given lines:

line #1: $\begin{cases} x = -1 + 3t \\ y = 3 - 2t \\ z = 1 + t \end{cases}$ **line #2:** $\begin{cases} x = 4 + 5t \\ y = 2 - t \\ z = -1 - 2t \end{cases}$

(c) Find a vector orthogonal to the plane given by $2x - 3y + z = 11$.

5. Determine the parametric equations for the line passing through the points (–3, 2, 0) and (4, 2, 3).

6. Three forces with magnitudes 3, 4, and 5 pounds act on a machine part at angles of –15°, 150°, and 220° respectively, with the positive *x*-axis. Find both the magnitude and direction of the resultant force. Round all answers to the nearest tenths of a unit. Be sure and include units in your final answer to get full credit.

7. Consider the following plane curves. Eliminate the parameter and represent each curve by a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

(a) $x = y^2 - 1$

(b) $y = x^2 - 1$

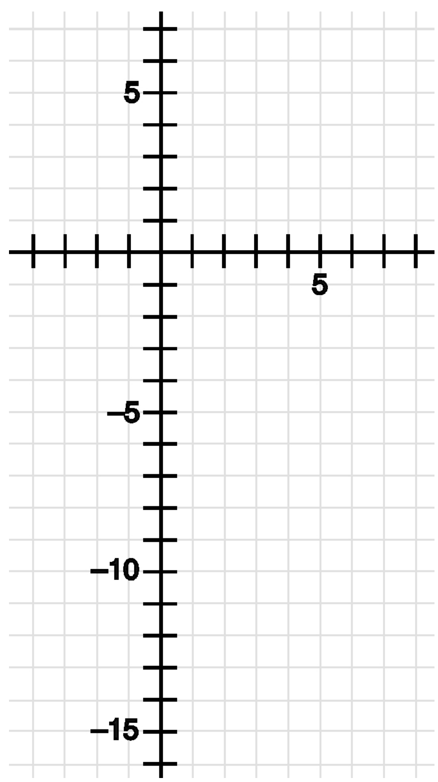
8. Find the domain of the vector-valued function $\mathbf{r}(t) = \left\langle \ln(3-t),\ \frac{\sqrt[4]{2+t}}{\ln|t|},\ \frac{t+1}{6t^2 - 7t - 3} \right\rangle$.
Write your answer answer using interval notation to get full credit.

9. Evaluate $\int \left\langle t\ln t,\ \sqrt{1+5t},\ \frac{t}{t+1} \right\rangle dt$.

10. An object starts from rest at the point (0, 1, 1) and moves with an acceleration $\mathbf{a}(t) = \langle 1, \cos t, 0 \rangle$. Find the position, $\mathbf{r}(t)$, at time $t = 4$.

11. Given the vector-valued function $\mathbf{r}(t) = \langle t+4, 1-t^2 \rangle$ ...

(a) Sketch the graph, be sure and identify all intercepts.



(b) Evaluate the velocity vector when $t = 2$, and sketch it on the same graph at the appropriate position.

12. ***Projectile Motion.*** A projectile is fired at a height of 2 meters above the ground with an initial velocity of 100 meters per second at an angle of 35° with the horizontal. Round each result to the nearest tenths of a unit.

(a) Find the vector-valued function describing the motion. ***Hint:*** Use $g = 9.8$ meters per second per second.

(b) Find the maximum height.

(c) How long was the projectile in the air?

(d) Find the range.

13. Find the unit tangent vector, $\mathbf{T}(t)$, for the curve given by $\mathbf{r}(t) = \langle 4\cos t, -3\sin t, 1\rangle$ when $t = \dfrac{3\pi}{2}$.

14. Find the length of the curve $\mathbf{r}(t) = \left\langle e^{t}, e^{-t}, \sqrt{2}t \right\rangle$ when $t \in [0, 2]$.

15. ***Domain for a function of 2 variables.*** Find the domain for the given function and write the answer using set notation:

$$f(x,y) = \frac{\ln(x-y)}{e^{1/x}} + \sin(x-y^2) + \sqrt[3]{x+3y} + \sqrt{x^2+2y^3} + \frac{x}{6x-7y}$$

16. Find the second partial, f_{xy}, for $f(x,y) = x^2y + 2y^2x^2 + 4x$.

17. Find the first partial derivative with respect to *x*: $F(x,y,z) = xe^{xyz}$

18. Use the total differential *dz* to approximate the change in $z = \dfrac{y}{x}$ as (*x*, *y*) moves from the point (2, 1) to the point (2.1, 0.8). Then, calculate the actual change Δz.

19. The radius of a right circular cylinder is decreasing at the rate of 4 inches per minute and the height is increasing at the rate of 8 inches per minute. What is the rate of change of the volume when r = 4 inches and h = 8 inches? ***(Hint:*** Use the Chain Rule for function of several variables.)

20. Find the directional derivative of $f(x,y) = x^2y$ at the point (1, –3) in the direction < –2, 1>.

21. Given the surface $2x^2 + 3y^2 + 4z^2 = 18 \ldots$

(a) Use implicit differentiation and find the slope in the x-direction, $\frac{\partial z}{\partial x}$, at the point (–1, 2, 1).

(b) Find an equation of the tangent plane (in general form) to the surface at the point (–1, 2, 1).

22. Find a set of parametric equations for the normal line to the surface given by $z = f(x, y) = x^2 y$ at the point (2, 1, 4).

23. Find extrema and saddle point(s), if any, for the function $f(x, y) = x^2 + x - 3xy + y^3 - 5$. Write your answer(s) in ordered triple(s) to get full credit.

24. Use Lagrange Multipliers to find the dimensions of a rectangular box of maximum volume with one vertex at the origin and the opposite vertex lying in the plane given by $6x + 4y + 3z = 24$. Then, give the actual maximum volume.

25. **(a)** Evaluate the integral: $\int_0^{\ln 2}\int_{e^y}^{2} x\,dxdy$

(b) Evaluate the integral $\int_0^1\int_y^1 x^2 e^{xy}\,dxdy$. ***(Hint:*** Reverse the order of integration first.)

26. Evaluate the double integral $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx$ by changing to polar coordinates.

27. Set up the triple integrals (but do not evaluate) that would calculate the volume of the solid bounded by the graphs of $z = 0$, $x^2 + y^2 = 16$, and $z = 5 - y$ using…

(a) rectangular coordinates

(b) cylindrical coordinates

28. Find work done by the force $\mathbf{F} = \langle y - x^2, z - y^2, x - z^2 \rangle$ over the curve $r(t) = \langle t, t^2, t^3 \rangle$ from the point (0, 0, 0) to (1, 1, 1).

29. Evaluate , where $C = \int_C \frac{x + y^2}{\sqrt{1 + x^2}} ds$. The curve is the straight line segment from the point (1, 0) to the point $\left(1, \frac{1}{2}\right)$, and $y = \frac{x^2}{2}$ is the curve along the graph of $\left(1, \frac{1}{2}\right)$ from the point to (0, 0).

30. Given the field $\mathbf{F} = \left\langle 2x, -y^2, -\frac{4}{1+z^2} \right\rangle \ldots$

(a) Show that the field is conservative.

(b) Evaluate $\int_{(0,0,0)}^{(3,3,1)} 2x dx - y^2 dy - \frac{4}{1+z^2} dz$.

31. Find the work done by the field $\mathbf{F} = \left\langle 2\cos y, \frac{1}{y} - 2x\sin y \right\rangle$ on the object that follows a path from the point (2, 1), to the point (2, 2), and then to the point $\left(1, \frac{\pi}{2}\right)$.

32. Use Green's Theorem to evaluate the line integral $\int_C \left(x - y^3\right)dx + x^3 dy$, where C is the right half of a circle of radius 2, $x^2 + y^2 = 4$.

33. Evaluate the surface integral $\int\int_S y\ dS$ if S is the part of the plane $z = 6 - 3x - 2y$ in the first octant.

34. Find the flux integral $\int_S\int \mathbf{F}\cdot\mathbf{N}\,dS$ if $\mathbf{F} = \langle x, y, z\rangle$ where S is the surface $z = 1 - x^2 - y^2$ above the xy-plane.

35. Let Q be the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$, and let $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2\rangle$. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}dS$. Source, sink, or neither?

36. Let Q be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{N}dS$ and calculate the outward flux of $\mathbf{F}$ through S, where S is the surface of Q and $\mathbf{F}(x,y,z) = \langle x,y,z\rangle$. Source, sink, or neither?

37. Let Q be the region bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and below by the plane $z = 0$ in the first octant. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$ and find the outward flux of $\mathbf{F}$ through S, where S is the surface of the solid and $\mathbf{F}(x, y, z) = \langle xy, 4x, 2y \rangle$. Source, sink, or neither?

38. Find the curl of the vector field $\mathbf{F}(x, y, z) = \langle z^2, -x^2, y \rangle$. Is the field conservative?

39. Use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle z, 2x, 2y \rangle$ and S is the surface of the paraboloid (oriented upward) of $z = 4 - x^2 - y^2$, $z \geq 0$, and C is its boundary.

40. Limits: Each of the following problems requires knowledge of limits.

(a) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{t\to 0}\left\langle \frac{\sin t - t}{t^3}, \frac{3t}{t-1}, \arctan\left(\frac{e^{t^2}}{2t^2}\right)\right\rangle$$

(b) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{t\to\infty}\left\langle e^{-t}, \frac{-t^2-2}{3t^2+5}, \frac{t^2}{\ln t}\right\rangle$$

(c) Find the limit at the boundary point (if it exists):

$$\lim_{(x,y)\to(3,2)} \frac{6x^2-13xy+6y^2}{6x^2-xy-12y^2}$$

(d) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{\sin(x^2+y^2+z^2)}$$

(e) Find the limit (if it exists). If it does not exist, so state. ***Hint:*** Convert to polar coordinates and use the fact that $(x,y)\to(0,0)$ means the same as $r\to 0$ along all paths in the domain to the point (0, 0).

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2y^2}{x^2+y^2}$$

(f) True or False:

The improper integral $\int_0^\infty \int_0^x \left(\frac{1}{x^2+1}\right)\left(\frac{1}{y^2+1}\right) dy dx$ converges to the value $\frac{\pi^2}{4}$

CALCULUS III — PRACTICE FINAL EXAM KEY

1. (a) The vector v has magnitude 8 and direction $\theta = 120°$. Find its component form.

Formula says: $\mathbf{v} = \langle \|\mathbf{v}\|\cos\theta, \|\mathbf{v}\|\sin\theta \rangle = \langle 8\cos 120°, 8\sin 120° \rangle = \left\langle 8\left(-\frac{1}{2}\right), 8\left(\frac{\sqrt{3}}{2}\right) \right\rangle = \langle -4, 4\sqrt{3} \rangle$ *(answer)*

(b) Suppose this same vector has as its initial point $(-2, 7\sqrt{3})$. Use your answer from part (a) to find its terminal point.

Always use "terminal minus initial point" as your mantra! So, if the terminal point has coordinates (*x, y*), then we need $x - (-2) = -4$, and we need $y - 7\sqrt{3} = 4\sqrt{3}$.

Solve for both *x* and *y* to get the point:

$$(-6, 11\sqrt{3}) \quad \textit{(answer)}$$

2. Determine if the following pairs of vector are orthogonal, parallel, or neither. Show your work.

(a) v = <3, –2> and w = <–1, 2>

(b) v = < –2, 0> and w = <0, 5>

(c) v = < –1, 2> and w = $\left\langle 0, -\frac{1}{2} \right\rangle$

(d) v = < 2, –3> and w = <–2, 3>

If they are orthogonal, then the dot product will be equal to zero. If they are parallel, then one must be a multiple of the other. That is, if **u** is parallel to **v**, then **u** = *c***v**. The pair in **(b)** is orthogonal, since the dot product is zero. (Try it!) The pair in **(d)** is parallel since **v** = –**w**. The other pairs are "neither."

3. Given the vectors u = < 2, –1, 1> and w = <–3, 2, 2>…

(a) Calculate the angle (in degrees) between the vectors. Round your answer to the nearest hundredth.

Use the formula $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|} = \frac{\langle 2,-1,1\rangle \cdot \langle -3,2,2\rangle}{\sqrt{2^2+1^2+1^2}\sqrt{3^2+2^2+2^2}} = \frac{-6-2+2}{\sqrt{6}\sqrt{17}} = -\frac{6}{\sqrt{102}}$

$$\text{So, } \theta = \cos^{-1}\left(\frac{-6}{\sqrt{102}}\right) \approx 126.45° \quad \textit{(answer)}$$

(b) Find $\text{proj}_{\mathbf{u}}\mathbf{w} = \left(\frac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|^2}\right)\mathbf{u} = \left(\frac{-6}{(\sqrt{6})^2}\right)\mathbf{u} = (-1)\mathbf{u} = -\langle 2,-1,1\rangle = \langle -2,1,-1\rangle$ *(answer)*

4. *Orthogonal Vectors:*

(a) Find a vector orthogonal to the *yz*-plane.

Answer: The standard unit vector $\mathbf{i} = \langle 1,0,0 \rangle$

(b) Find a vector orthogonal to the two given lines:

$$\text{line \#1: } \begin{cases} x = -1+3t \\ y = 3-2t \\ z = 1+t \end{cases} \qquad \text{line \#2: } \begin{cases} x = 4+5t \\ y = 2-t \\ z = -1-2t \end{cases}$$

The vectors $\mathbf{v}_1 = \langle 3,-2,1 \rangle$ and $\mathbf{v}_2 = \langle 5,-1,-2 \rangle$ are the direction vectors for the lines, respectively.

ANSWER 4, CONTINUED...

The cross-product of these two vectors will be a vector orthogonal to both vectors, and hence, both:

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 5 & -1 & -2 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 5 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 5 & -1 \end{vmatrix}\mathbf{k} = (4+1)\mathbf{i} - (-6-5)\mathbf{j} + (-3+10)\mathbf{k} = \langle 5,11,7 \rangle \quad \textit{(answer)}$$

(c) Find a vector orthogonal to the plane given by $2x - 3y + z = 11$.

Very simply, the coefficients of the variables in the equation for a plane give the components for the vector normal to the plane $\mathbf{v} = \langle 2,-3,1 \rangle$ *(answer)*

5. Determine the parametric equations for the line passing through the points (–3, 2, 0) and (4, 2, 3).

$\overrightarrow{PQ} = \text{terminal} - \text{initial} = \langle 4-(-3),\, 2-2,\, 3-0 \rangle = \langle 7,\, 0,\, 3 \rangle$ This is the direction vector for the line. The parametric equations for a line are $x = x_1 + at,\ y = y_1 + bt,\ z = z_1 + ct$, where the direction vector is $\mathbf{v} = \langle a,b,c \rangle$. Use either point to substitute into these equations for $P(x_1, y_1, z_1)$. I'll use the first point:

$$x = -3 + 7t,\ y = 2,\ z = 3t \quad \textit{(answer)}$$

6. Three forces with magnitudes 3, 4, and 5 pounds act on a machine part at angles of –15°, 150°, and 220° respectively, with the positive *x*-axis. Find both the magnitude and direction of the resultant force. Round all answers to the nearest tenths of a unit. Be sure and include units in your final answer to get full credit.

First, we need to write all three forces in component form:

$$\mathbf{F}_1 = \langle 3\cos(-15°), 3\sin(-15°) \rangle \approx \langle 2.90, -0.78 \rangle$$

$$\mathbf{F}_2 = \langle 4\cos(150°), 4\sin(150°) \rangle \approx \langle -3.46, 2 \rangle$$

$$\mathbf{F}_3 = \langle 5\cos(220°), 5\sin(220°) \rangle \approx \langle -3.83, -3.21 \rangle$$

Next, we sum all of the forces to get the resultant vector:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \langle 2.90 + (-3.46) + (-3.83), -0.78 + 2 + (-3.21) \rangle \approx \langle -4.39, -1.99 \rangle$$

Note that this resultant vector is in Quadrant III. This is important, especially when we try and find the direction (angle) that it makes with the positive *x*-axis. The magnitude of this resultant vector is given by:

$$\|\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\| = \sqrt{(-4.39)^2 + (-1.99)^2} \approx 4.8\ lbs.$$

To find the direction, we first take the inverse tangent of the quotient of the *y*-component divided by the *x*-component:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1.99}{-4.39}\right) \approx 24.4°$$

This is not correct, since the angle of 24.4° is in Quadrant I. The reason why our calculator gives us this answer is because the range of the arctangent function lies in the interval $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. To get the correct angle (lying in Quadrant III), we add 180° to our result to get:

$$\text{Direction} = \text{angle} = \theta = 24.4° + 180° = 204.4°$$

Final answer: The magnitude is 4.8 pounds and the direction is 204.4°

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

7. Consider the following plane curves. Eliminate the parameter and represent each curve by a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

(a) $x = y^2 - 1$

Let $y = t$, then $x = t^2 - 1$.

So, $\mathbf{r}(t) = \langle t^2 - 1, t \rangle$ *(answer)*

Caution! Do *not* use $x = t$, because then there will be two answers for y:
$y = \pm\sqrt{x+1}$
(and that's not very "nice.")

(b) $y = x^2 - 1$

Let $x = t$, then $y = t^2 - 1$, so we have the vector-valued function $\mathbf{r}(t) = \langle t, t^2 - 1 \rangle$.

8. Find the domain of the vector-valued function $\mathbf{r}(t) = \left\langle \ln(3-t), \dfrac{\sqrt[4]{2+t}}{\ln|t|}, \dfrac{t+1}{6t^2 - 7t - 3} \right\rangle$.

Write your answer answer using interval notation to get full credit.

The first component contains a logarithm, so we "need to worry." The argument of the logarithm must be positive, so we require that:

$$3 - t > 0$$
$$3 > t$$
$$or:\ t < 3$$

The second component has two "issues." First, we need to guarantee that the radicand of the even root is never negative. In other words $2 + t \geq 0$, or $t \geq -2$.

Also, we never want the denominator to be equal to zero. The denominator $\ln|t|$ will not equal zero so long as the argument t is never equal to 1. That is, since $\ln|\pm 1| = 0$, we want $t \neq \pm 1$. Further, we need the argument to be non-zero, so $t \neq 0$ also. We do *not* need to worry about this particular argument being negative as we did for the logarithm in the first component because of the absolute value symbol. In other words, this logarithm is OK with negative numbers since they will "automatically" become positive due to the absolute value.

Finally, the last component has a denominator we need to worry about becoming zero. First we set the denominator equal to zero, solve the quadratic equation, and then eliminate the solutions for this equation from the domain. We will solve by factoring:

$$6t^2 - 7t - 3 = 0$$
$$6t^2 + 2t - 9t - 3 = 0$$
$$2t(3t+1) - 3(3t+1) = 0$$
$$(2t-3)(3t+1) = 0$$
$$2t - 3 = 0 \quad or \quad 3t + 1 = 0$$
$$t = \frac{3}{2}, \qquad t = \frac{-1}{3}$$

ANSWER 8, CONTINUED...

This means we exclude the solutions from our domain. So we have $t \neq \frac{3}{2},\ t \neq \frac{-1}{3}$.

Finally, we intersect all four of our sets of domains for individual functions, and we have the final result:

$$t \in [-2, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 0\right) \cup (0, 1) \cup \left(1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 3\right) \textit{ (answer)}$$

9. Evaluate $\int \left\langle t\ln t,\ \sqrt{1+5t},\ \frac{t}{t+1} \right\rangle dt$.

This is an indefinite integral, so the final answer will have a constant *vector* of ***C***. If it were a definite integral, we would evaluate each anti-derivative at the limits.

The integral $\int t\ln t\,dt$ requires integration by parts, where $u = \ln t,\ dv = t\,dt,\ du = \frac{1}{t}dt,\ v = \frac{t^2}{2}$.

Then, we have $\int t\ln t\,dt = uv - \int v\,du = (\ln t)\left(\frac{t^2}{2}\right) - \int \left(\frac{t^2}{2}\right)\left(\frac{1}{t}\right)dt = \frac{t^2 \ln t}{2} - \frac{1}{2}\int t\,dt = \frac{t^2 \ln t}{2} - \frac{t^2}{4} + C_1$.

The integral $\int \sqrt{1+5t}\,dt$ can be evaluated using the general power rule, where $u = 1+5t,\ \ du = 5dt$.

So, we have $\int \sqrt{1+5t}\,dt = \frac{1}{5}\int \sqrt{1+5t}(5)dt = \frac{1}{5}\int \sqrt{u}\,du = \frac{1}{5}\int u^{1/2}du = \frac{1}{5}\left(\frac{2}{3}\right)u^{3/2} = \frac{2}{15}(1+5t)^{3/2} + C_2$.

The last integral has an improper rational function in its integrand, and so requires polynomial long division first:

$$t+1\overline{\smash{)}\,t}\quad \text{quotient } 1;\quad t - (t+1) = -1$$

Write in the form of:

$$\textit{Quotient} + \frac{\textit{rem}}{\textit{divisor}} = 1 + \frac{-1}{t+1}$$

The integral now looks like $\int \left(1 - \frac{1}{t+1}\right)dt = t - \ln|t+1| + C_3$.

The final answer must be written as a vector:

$$\left\langle \frac{t^2 \ln t}{2} - \frac{t^2}{4},\ \frac{2}{15}(1+5t)^{3/2},\ t - \ln|t+1| \right\rangle + \langle C_1, C_2, C_3 \rangle$$

Or, you can write the constant vector differently:

$$\left\langle \frac{t^2 \ln t}{2} - \frac{t^2}{4},\ \frac{2}{15}(1+5t)^{3/2},\ t - \ln|t+1| \right\rangle + \mathbf{C} \quad \textit{(answer)}$$

10. An object starts from rest at the point (0, 1, 1) and moves with an acceleration $\mathbf{a}(t) = \langle 1, \cos t, 0 \rangle$. Find the position, r(*t*), at time *t* = 4.

Starting from rest means the initial velocity is $\mathbf{v}(0) = \langle 0, 0, 0 \rangle$. We find velocity first by integrating acceleration:

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int \langle 1, \cos t, 0 \rangle dt = \langle t, \sin t, 0 \rangle + \langle C_1, C_2, C_3 \rangle$$

To find the constants, we use the initial condition for velocity:

$$\mathbf{v}(0) = \langle 0, 0, 0 \rangle = \langle 0 + C_1, \sin(0) + C_2, 0 + C_3 \rangle$$

We equate the components and find that $C_1 = 0, C_2 = 0, C_3 = 0$.

Now, we have the complete velocity function $\mathbf{v}(t) = \langle t, \sin t, 0 \rangle$.

We integrate velocity now to find the position function:

$$\mathbf{r}(t)=\int \mathbf{v}(t)dt=\int\langle t,\sin t,0\rangle dt=\left\langle \frac{t^2}{2}+C_{11},-\cos t+C_{22},0+C_{33}\right\rangle$$

We use the initial condition for position in order to find the new sets of constants of integration:

$$\mathbf{r}(0)=\langle 0,1,1\rangle=\left\langle \frac{0^2}{2}+C_{11},-\cos(0)+C_{22},C_{33}\right\rangle$$

Equating the components for these vectors gives us that $C_{11}=0, C_{22}=2, C_{33}=1$.

Our final answer is now $\mathbf{r}(t)=\left\langle \frac{t^2}{2},-\cos(t)+2,1\right\rangle$

11. Given the vector-valued function $\mathbf{r}(t)=\langle t+4,1-t^2\rangle$ **...**

(a) Sketch the graph, be sure and identify all intercepts.

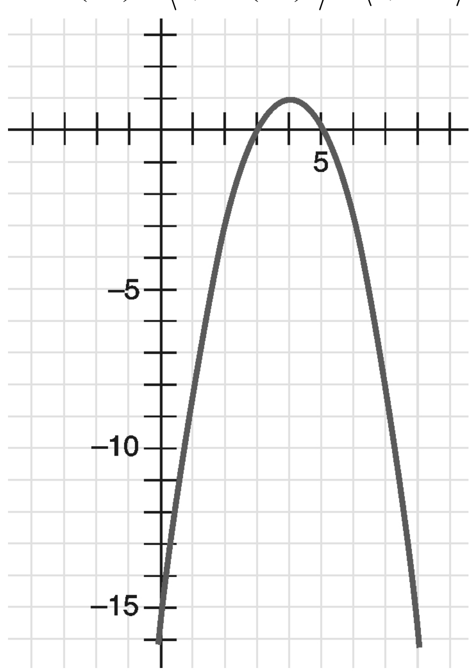
The easiest way to graph this vector-valued function is to use your graphing calculator in parametric mode. Another way is to make a table with two columns: one for time *t*, and the other for the corresponding ordered pair, (*x*, *y*).

The *x*-intercept(s) can be found by setting the *y*-component equal to 0, and solving for *t*:

$$1-t^2=0$$
$$(1+t)(1-t)=0$$
$$t=\pm 1$$

The position for when $t=\pm1$ occurs when the vector-valued function is evaluated at these two points: $\mathbf{r}(1)=\langle 1+4,0\rangle$ and $\mathbf{r}(-1)=\langle -1+4,0\rangle$. That is, at the points (5, 0) and (3,0).

The *y*-intercept(s) can be found by setting the *x*-component equal to 0: $t+4=0$. So we have $t=-4$. This occurs at the position $\mathbf{r}(-4)=\langle 0,1-(-4)^2\rangle=\langle 0,-15\rangle$. So, the y-intercept occurs at the point (0, –15).



Answer 11, continued...

(b) Evaluate the velocity vector when $t =$ 2, and sketch it on the same graph at the appropriate position.

First we find the velocity vector, which is the first derivative of the vector-valued function $\mathbf{r}'(t) = \langle 1, -2t \rangle$.

Next, we evaluate this at the specified time, $t = 2$: $\mathbf{r}'(2) = \langle 1, -2(2) \rangle = \langle 1, -4 \rangle$. We sketch this vector at the appropriate position. This position is found by evaluating the position vector given, $\mathbf{r}(t)$ at the time $t = 2$. That is, the position is: $\mathbf{r}(2) = \langle 2+4, 1-(2)^2 \rangle = \langle 6, -3 \rangle$.

So, sketch the vector <1, –4> where the initial point of the vector is located at the point (6, –3) on the graph. ***Remember:*** velocity vectors are always *tangent* to the line of motion. So, if your velocity vector does not appear to be tangent to the curve, you probably did something wrong!

12. ***Projectile Motion*. A projectile is fired at a height of 2 meters above the ground with an initial velocity of 100 meters per second at an angle of 35° with the horizontal. Round each result to the nearest tenths of a unit.**

(a) Find the vector-valued function describing the motion. *Hint:* Use g = 9.8 meters per second per second.

Use the Projectile Motion Theorem $\mathbf{r}(t) = \left\langle v_0 \cos\theta\, t,\ h + v_0 \sin\theta\, t - \frac{1}{2} g t^2 \right\rangle$.

So using $v_0 = 100 m/\text{sec}, h = 2m,\ g = 9.8 m/\text{sec}^2$, $\theta = 35°$, we have:

$$\mathbf{r}(t) = \left\langle 100\cos(35°)\,t,\ 2 + 100\sin(35°)\,t - \frac{1}{2}(9.8)t^2 \right\rangle \approx \langle 81.915t,\ 2 + 57.358t - 4.9t^2 \rangle$$

(b) Find the maximum height.

Maximum height occurs when vertical component of velocity is equal to zero. The velocity vector is given by $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 81.915,\ 57.358 - 9.8t \rangle$. Set the vertical component equal to zero. Be careful! The solution for this equation will give the *time* it takes for the projectile to achieve its maximum height, and *not* the actual maximum height. You then plug this value for time into the vertical component of the position function to obtain the maximum height:

$$57.358 - 9.8t = 0$$

$$t = \frac{57.358}{9.8} \approx 5.85 \text{ seconds}$$

Please note that you *do not* want to round to tenths just yet. Wait until the final result to round your answer to the nearest tenths, else your answer will not be as accurate.

Then, plug t = 5.85 seconds into the y-component of $\mathbf{r}(t)$ to get:

$$2 + 57.35(5.85) - 4.9(5.85)^2 \approx 169.8 \text{ meters}$$

(c) How long was the projectile in the air?

The projectile will hit the ground and end its journey when height = 0. So, we will set the vertical component of the position function equal to zero and solve for time t. This will give us how long the projectile was in the air.

Set the y-component of $\mathbf{r}(t)$ equal to zero to get $2 + 57.35t - 4.9t^2 = 0$.

This is a quadratic equation. We will solve it using the quadratic formula, where

$t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = -4.9$, $b = 57.35$, and $c = 2$.

We get two solutions: $t = 11.739$ seconds and $t = -0.034$ seconds (which is impossible here).

Answer: The projectile was in the air for a total of time $t = 11.739$ seconds.

(d) Find the range.

To get the range, we plug our result from part (c) into the horizontal component of the position function to get (81.915)(11.739) which is approximately equal to 961.6 meters. *(answer)*

13. Find the unit tangent vector, T(*t*), for the curve given by $\mathbf{r}(t) = \langle 4\cos t, -3\sin t, 1\rangle$ **when** $t = \dfrac{3\pi}{2}$**.**

We use the definition:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -4\sin t, -3\cos t, 0\rangle}{\sqrt{(-4\sin t)^2 + (-3\cos t)^2 + (0)^2}} = \frac{\langle -4\sin t, -3\cos t, 0\rangle}{\sqrt{16\sin^2 t + 9\cos^2 t}}$$

Evaluating at time $t = \dfrac{3\pi}{2}$, we have that:

$$\mathbf{T}\left(\frac{3\pi}{2}\right) = \frac{\langle -4\sin(3\pi/2), -3\cos(3\pi/2), 0\rangle}{\sqrt{16\sin^2(3\pi/2) + 9\cos^2(3\pi/2)}} = \frac{\langle (-4)(-1), (-3)(0), 0\rangle}{\sqrt{16(1) + 9(0)}} = \frac{\langle 4, 0, 0\rangle}{\sqrt{16}} = \langle 1,0,0\rangle \quad \textit{(answer)}$$

14. Find the length of the curve $\mathbf{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t\rangle$**, when [0, 2].**

Use the formula for arclength:

$$s = \int_a^b \|\mathbf{r}'(t)\| dt = \int_0^2 \left\|\langle e^t, -e^{-t}, \sqrt{2}\rangle\right\| dt = \int_0^2 \sqrt{(e^t)^2 + (-e^{-t})^2 + (\sqrt{2})^2}\, dt = \int_0^2 \sqrt{e^{2t} + e^{-2t} + 2}\, dt = \int_0^2 \sqrt{e^{2t} + \frac{1}{e^{2t}} + 2}\, dt = \int_0^2 \sqrt{\frac{e^{4t} + 1 + 2e^{2t}}{e^{2t}}}\, dt$$

The numerator in the radicand factors:

$$\int_0^2 \sqrt{\frac{(e^{2t}+1)^2}{e^{2t}}}\, dt = \int_0^2 \frac{e^{2t}+1}{e^t}\, dt = \int_0^2 \left(\frac{e^{2t}}{e^t} + \frac{1}{e^t}\right) dt = \int_0^2 (e^t + e^{-t})\, dt$$

We integrate now:

$$\left[e^t - e^{-t}\right]_0^2 = (e^2 - e^{-2}) - (e^0 - e^0) = e^2 - e^{-2}$$

(answer)

15. ***Domain for a function of 2 variables.*** **Find the domain for the given function and write the answer using set notation:**

$$f(x, y) = \frac{\ln(x-y)}{e^{1/x}} + \sin(x - y^2) + \sqrt[3]{x + 3y} + \sqrt{x^2 + 2y^3} + \frac{x}{6x - 7y}$$

Answer: $\{(x, y) \mid x - y > 0, x \neq 0, x^2 + 2y^3 \geq 0, 6x - 7y \neq 0\}$

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

16. **Find the second partial, f_{xy}, for $f(x,y) = x^2y + 2y^2x^2 + 4x$.**

We first find the first partial: $f_x = 2xy + 4xy^2 + 4$.

To find the second partial, f_{xy}, we integrate the first partial w.r.t. *y* and get:

$$f_{xy} = 2x + 8xy \ \textit{(answer)}$$

17. **Find the first partial derivative with respect to *x*: $F(x,y,z) = xe^{xyz}$.**

We need to use the Product Rule, and take the derivative with respect to *x*. This means that we treat both the variables *y* and *z* as constants:

$$F_x(x,y,z) = (1)e^{xyz} + x(yz)e^{xyz} = e^{xyz}(1 + xyz)$$

18. **Use the total differential *dz* to approximate the change in $z = \frac{y}{x}$ as (*x*, *y*) moves from the point (2, 1) to the point (2.1, 0.8). Then, calculate the actual change Δz.**

First, we find the total differential that is given by $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$.

So, we find the partial derivatives first: $\frac{\partial z}{\partial x} = (-1)yx^{-2} = -\frac{y}{x^2}$ and $\frac{\partial z}{\partial y} = \frac{1}{x}$.

Next, we note that $dx = \Delta x = 2.1 - 2 = 0.1$ and also that $dy = \Delta y = 0.8 - 1 = -0.2$.

Then we have the total differential evaluated at the starting point of (2,1):

$$dz = -\frac{1}{(2)^2}(0.1) + \frac{1}{2}(-0.2) = -0.125 = -\frac{1}{8}$$

The actual change for *z* if given by:

$$\Delta z = f(2.1, 0.8) - f(2,1) = \frac{0.8}{2.1} - \frac{1}{2} = -0.1190476 = -\frac{5}{42} \text{ (A pretty close approximation!)}$$

19. **The radius of a right circular cylinder is decreasing at the rate of 4 inches per minute and the height is increasing at the rate of 8 inches per minute. What is the rate of change of the volume when r = 4 inches and h = 8 inches? *(Hint:* Use Chain Rule for function of several variables.)**

The volume of a right circular cylinder is a function of two variables: $V(r,h) = \pi r^2 h$.

Using the Chain Rule for Functions of Several Variables, we differentiate both sides of this "equation" with respect to time *t*:

$$\frac{d[V(r,h)]}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

Taking all of the partial derivatives as well as the "regular" ones, we find that:

$$\frac{dV}{dt} = (2\pi hr)\cdot\frac{dr}{dt} + (\pi r^2)\cdot\frac{dh}{dt}$$

Now, we use all of the info given, that is:

$$\frac{dr}{dt} = -4in/\min,\ \frac{dh}{dt} = 8in/\min,\ r = 4in,\ h = 8in$$

$$\frac{dV}{dt} = (2\pi)(8)(4)\cdot(-4) + \pi(4)^2\cdot(8) = -128\pi \text{ cubic inches per minute } \textit{(answer)}$$

Calculus III — Practice Final Exam Key, continued

20. Find the directional derivative of $f(x,y) = x^2 y$ at the point (1, –3) in the direction < –2, 1>.

We first need to normalize the direction vector. That is, form a unit vector out of the direction vector:

$$\mathbf{u} = \frac{\langle -2,1\rangle}{\sqrt{2^2+1^2}} = \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

Next, we find the first partials for the function: $f_x = 2xy,\ f_y = x^2$.

Then, we evaluate these partials at the point given, that is (1, –3):

$$f_x\big|_{(1,-3))} = 2(1)(-3) = -6, \quad f_y\big|_{(1,-3)} = (1)^2 = 1$$

We form the gradient now:

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xy, x^2 \rangle$$

At the point (1, –3), the gradient is the vector $\nabla f(1,-3) = \langle -6,1 \rangle$.

Finally, we use the definition for the directional derivative that is:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \langle \cos\theta, \sin\theta \rangle = \langle -6,1 \rangle \cdot \left\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = (-6)\left(\frac{-2}{\sqrt{5}}\right) + (1)\left(\frac{1}{\sqrt{5}}\right) = \frac{13}{\sqrt{5}} \quad \textit{(answer)}$$

Note: this result gives the *slope* of the surface in the direction of the direction vector!

21. Given the surface $2x^2 + 3y^2 + 4z^2 = 18$ …

(a) Use implicit differentiation and find the slope in the *x*-direction, $\frac{\partial z}{\partial x}$, at the point (–1, 2, 1).

Use the definition $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, where $F(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 18 = 0$.

Therefore, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x}{8z}$.

Then at the point (–1, 2, 1), we have the slope as:

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,2,1)} = -\frac{4(-1)}{8(1)} = \frac{1}{2} \quad \textit{(answer)}$$

(b) Find an equation of the tangent plane (in general form) to the surface at the point (–1, 2, 1).

The gradient of the function $F(x,y,z)$ found in part **(a)** above will be a vector orthogonal to the surface that we will use to form the equation of the plane:

$$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle = \langle 4x, 6y, 8z \rangle$$

Evaluate at the point given: $\nabla F(-1,2,1) = \langle 4(-1), 6(2), 8(1) \rangle = \langle -4, 12, 8 \rangle$.

Now we use the format for the equation of a plane, which is: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where the vector orthogonal to the plane has components *a, b*, and *c*.

So, $-4(x-(-1)) + 12(y-(2)) + 8(z-(1)) = 0$.

Distribute and collect like terms to get the equation into general form:

$$-x + 3y + 2z = 9 \quad \textit{(answer)}$$

Calculus III — Practice Final Exam KEY, continued

22. **Find a set of parametric equations for the normal line to the surface given by $z = f(x, y) = x^2 y$ at the point (2, 1, 4).**

First rewrite the equation in the form $F(x, y, z) = 0$. We will do this by subtracting $x^2 y$ from both sides: $z - x^2 y = 0$.

We then find the gradient $\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z) \rangle$.

This gradient vector will be a vector that is normal to the surface. It will also be a vector that will serve as the direction vector for our normal line to the surface.

So, finding the first partials we obtain $\nabla F(x, y, z) = \langle -2xy, -x^2, 1 \rangle$.

We evaluate the gradient at the given point, (2, 1, 4):

$$\nabla F(2, 1, 4) = \langle -2(2)(1), -(2)^2, 1 \rangle = \langle -4, -4, 1 \rangle$$

Now we have all of the information we need to write the set of parametric equations for our line that is normal to the surface. We use where the direction vector is given by $\mathbf{v} = \langle a, b, c \rangle$ and a point on the line given by (x_0, y_0, z_0).

$$\Longrightarrow \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

The answer is: $\begin{cases} x = 2 - 4t \\ y = 1 - 4t \\ z = 4 + t \end{cases}$

23. **Find extrema and saddle point(s), if any, for the function $f(x, y) = x^2 + x - 3xy + y^3 - 5$. Write your answer(s) in ordered triple(s) to get full credit.**

We first find the first partials. Then we form a system of equations by setting both partial simultaneously equal to zero, and solve the system.

The solution(s) of this system give the critical point(s) $f_x = 2x + 1 - 3y = 0,\ f_y = -3x + 3y^2 = 0$.

This is a non-linear system, which is most easily solved using the method of substitution. Using the second equation, solve for *x*, then substitute into the first equation:

$$3y^2 = 3x \Rightarrow x = y^2 \Rightarrow 2(y^2) - 3y + 1 = 0$$

This is a quadratic equation. We can solve by either using the quadratic formula, or by factoring:

$$(2y-1)(y-1) = 0 \Rightarrow 2y - 1 = 0, \quad or \quad y - 1 = 0 \Rightarrow y = \frac{1}{2}, 1$$

Since $x = y^2$, we get the corresponding *x*-coordinates, and find that the solutions to the system, which are exactly the critical points $(1,1)$ *and* $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Now we form *D* for the *D*-Test: $D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (2)(6y) - (-3)^2 = 12y - 9$

We perform the *D*-test by evaluating *D* at each of the critical points: $D(1,1) = 12(1) - 9 = 3 > 0$.

Since $D > 0$, we have an extrema. To find out what type, we evaluate the second partial f_{xx} at the point $f_{xx}(1,1) = 2 > 0$.

Since it's also positive, we have a relative minimum at the point (1, 1, –5).

Answer 23, continued...

...ANSWER 23, CONTINUED

Now, test the other point by evaluating D:

$$D\left(\frac{1}{4},\frac{1}{2}\right)=12\left(\frac{1}{2}\right)-9=-3<0$$

We have a saddle point at $\left(\frac{1}{4},\frac{1}{2},-\frac{79}{16}\right)$. *(answer)*

24. Use Lagrange Multipliers to find the dimensions of a rectangular box of maximum volume with one vertex at the origin and the opposite vertex lying in the plane given by $6x+4y+3z=24$. Then, give the actual maximum volume.

The function we are trying to maximize is the volume function for a rectangular box. For this problem where one vertex is at the origin, we will have width = x, length = y, and height = z. So, the volume of the box is given by $f(x,y,z)=xyz$.

The constraint equation needs to be rewritten as $g(x,y,z)=0$. The plane equation is our constraint equation. We rewrite it $6x+4y+3z-24=0$.

Next, we form the Lagrange function:

$$L(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z)$$

Using our function and constraint, our Lagrange function looks like:

$$L(x,y,z,\lambda)=xyz-\lambda(6x+4y+3z-24)$$

Next, we find all four first partial derivatives for the Lagrange function. We set all four of these partials equal to zero, and then solve the resulting (non-linear in this case) system of equations:

$$L_x=yz-6\lambda=0$$
$$L_y=xz-4\lambda=0$$
$$L_z=xy-3\lambda=0$$
$$L_\lambda=-(6x+4y+3z-24)=0$$

Notice that the last partial is simply our constraint equation $g(x,y,z)=0$, if you go ahead and rewrite it. By the way, this will always be the case when using the method of Lagrange Multipliers for the last partial (with respect to lambda—λ).

There are several approaches in attempting to solve this system. The strategy l will use is the following: Solve for λ using the first three equations. Use these to get both y and z in terms of x. Substitute these into the constraint equation to find x:

$$\lambda=\frac{yz}{6}$$
$$\lambda=\frac{xz}{4}$$
$$\lambda=\frac{xy}{3}$$

ANSWER 24, CONTINUED...

...ANSWER 24, CONTINUED

CALCULUS III **Practice FINAL EXAM KEY, CONTINUED**

Setting the first form for l equal to the second one: $\text{l} = \frac{yz}{6} = \frac{xz}{4}$. Next divide both sides by z to get that $y = \frac{3}{2}x$.

Next, set the second and third equations equal to each other: $\frac{xz}{4} = \frac{xy}{3}$.

Then divide both sides by x to get that $z = \frac{4}{3}y = \frac{4}{3}\left(\frac{3}{2}x\right) = 2x$.

Substitute both $y = \frac{3}{2}x$ and $z = 2x$ into the constraint equation and solve for x:

$$6x + 4\left(\frac{3}{2}x\right) + 3(2x) = 24$$

$$6x + 6x + 6x = 24$$

$$18x = 24$$

$$x = \frac{24}{18} = \frac{4}{3}$$

Now, we get the other dimensions for the box:

$$y = \frac{3}{2}x = \left(\frac{3}{2}\right)\left(\frac{4}{3}\right) = 2$$

$$z = 2x = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

Therefore the dimensions are $x = \frac{4}{3},\ y = 2,\ z = \frac{8}{3}$.

The maximum volume for the box is $f\left(\frac{4}{3}, 2, \frac{8}{3}\right) = \left(\frac{4}{3}\right)(2)\left(\frac{8}{3}\right) = \frac{64}{9}$ cubic units. *(answer)*

25. (a) Evaluate the integral:

$$\int_0^{\ln 2}\int_{e^y}^{2} x\,dx\,dy = \int_0^{\ln 2}\left[\frac{x^2}{2}\right]_{e^y}^{2} dy = \int_0^{\ln 2}\left(\frac{2^2}{2} - \frac{(e^y)^2}{2}\right)dy = \int_0^{\ln 2}\left(2 - \frac{e^{2y}}{2}\right)dy = \left[2y - \frac{e^{2y}}{4}\right]_0^{\ln 2} = \left[2(\ln 2) - \frac{e^{2(\ln 2)}}{4}\right] - \left(2(0) - \frac{e^{(2)(0)}}{4}\right)$$

$$= 2\ln 2 - \frac{2^2}{4} + \frac{1}{4} = 2\ln 2 - \frac{3}{4} \quad \textit{(answer)}$$

Note: We used the fact that $e^{\ln x} = x$ in the last step.

ANSWER 25, CONTINUED...

...ANSWER 25, CONTINUED

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

(b) Evaluate the integral $\int_0^1\int_y^1 x^2 e^{xy}\,dxdy$. (*Hint:* Reverse order of integration first.)

We sketch the region of integration by graphing the equations $y=0,\ y=1,\ x=y,\ \textit{and}\ \ x=1$. It is a triangle with vertices: (0, 0), (1, 0), and (1, 1).

To switch the order of integration, we'll require the variable *x* to have constants as limits, so, $0 \le x \le 1$, and $0 \le y \le x$.

We have $\int_0^1\int_0^x x^2 e^{xy}\,dydx$.

To integrate, we use the exponential rule, where $u = xy$, and $du = xdx$. Save one factor of x^2 for the *u*-substitution, and the other we'll treat as a constant multiple:

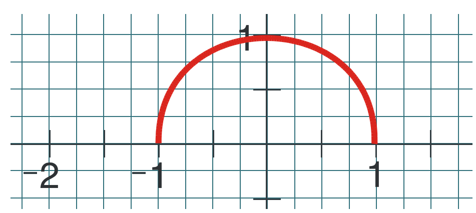
$$\int_0^1 x\int_0^x xe^{xy}\,dydx = \int_0^1 x\left[\int e^u\,du\right]dx = \int_0^1 x\left[e^{xy}\right]_0^x dx = \int_0^1 x\left[e^{x^2} - e^0\right]dx = \int_0^1 \left(xe^{x^2} - x\right)dx.$$

We'll use *u*-substitution again with the exponential rule, where $u = x^2$, $du = 2xdx$. Now, we have:

$$\frac{1}{2}\int_0^1 2xe^{x^2}\,dx - \int_0^1 xdx = \left[\frac{1}{2}e^{x^2} - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}e - \frac{1}{2} - \left(\frac{1}{2}e^0 - 0\right) = \frac{1}{2}e - 1 \quad (\textit{answer})$$

26. Evaluate the double integral $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dydx$ by changing to polar coordinates.

First, we'll sketch the region of integration in order to be able to see what the limits for both *r* and θ will be. We see that $0 \le \theta \le \pi$, and that $0 \le r \le 1$.



$$\int_0^\pi\int_0^1 e^{r^2}\,rdrd\theta = \frac{1}{2}\int_0^\pi\int_0^1 e^{r^2}\,2rdrd\theta = \frac{1}{2}\int_0^\pi\left[e^{r^r}\right]_0^1 d\theta = \frac{1}{2}\int_0^\pi (e-1)\,d\theta$$

$$= \frac{1}{2}(e-1)\left[\theta\right]_0^\pi = \frac{1}{2}(e-1)\pi \quad (\textit{answer})$$

27. Set up the triple integrals (do not evaluate) that would calculate the volume of the solid bounded by the graphs of $z = 0$, $x^2 + y^2 = 16$, and $z = 5 - y$ using…

(a) rectangular coordinates

$$V = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_0^{5-y} dzdydx \quad (\textit{answer})$$

(b) cylindrical coordinates

$$V = \int_0^{2\pi}\int_0^4\int_0^{5-r\sin\theta} dzrdrd\theta \quad (\textit{answer})$$

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

28. Find work done by the force $\mathbf{F}(x, y, z) = \langle y - x^2, z - y^2, x - z^2 \rangle$ over the curve $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ from the point (0, 0, 0) to (1, 1, 1).

We cannot use the Fundamental Theorem of Line Integrals because the vector field is not conservative. Instead, we use $Work = \int_C \mathbf{F} \cdot d\mathbf{r}$, where we have that $d\mathbf{r} = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.

We then replace *x*, *y*, and *z* in the vector field with the components from $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, and have:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^2 - t^2, t^3 - (t^2)^2, t - (t^3)^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle dt = \int_0^1 \langle 0, t^3 - t^4, t - t^6 \rangle \cdot \langle 1, 2t, 3t^2 \rangle dt = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8) dt$$

Integrate and evaluate, and get: $= \left[\frac{2}{5}t^5 - \frac{1}{3}t^6 + \frac{3}{4}t^4 - \frac{1}{3}t^9 \right]_0^1 = \frac{29}{60}$ *(answer)*

29. Evaluate $\int_C \frac{x + y^2}{\sqrt{1 + x^2}} ds$, where $C = C_1 + C_2$.

The curve C_1 is the straight line segment from the point (1, 0) to the point $\left(1, \frac{1}{2}\right)$.

C_2 is the curve along the graph of $y = \frac{x^2}{2}$ from the point $\left(1, \frac{1}{2}\right)$ to (0, 0).

We first parameterize each of the curves: $C_1 : \mathbf{r}_1(t) = \langle 1, t \rangle$, $0 \le t \le \frac{1}{2}$.

So, the arclength element for this parameterization is $ds = \|\mathbf{r}_1'(t)\| dt = \sqrt{0^2 + 1^2} = 1dt$.

Parameterizing the second curve for the path, we have that:

$$C_2 : \mathbf{r}_2(t) = \left\langle 1 - t, \frac{(1-t)^2}{2} \right\rangle, \quad 0 \le t \le 1$$

Taking the derivative of $\mathbf{r}_2$, we get that:

$$\mathbf{r}_2'(t) = \left\langle -1, \frac{2(1-t)}{2}(-1) \right\rangle = \langle -1, t - 1 \rangle$$

Next, we calculate the arclength element for this parameterization:

$$ds = \|\mathbf{r}_2'(t)\| dt = \sqrt{(-1)^2 + (t-1)^2} dt = \sqrt{1 + (t-1)^2} dt$$

Then, we evaluate the line integral where we rewrite it as a sum of two separate integrals (with all of the appropriate replacements and substitutions:

$$\int_C \frac{x + y^2}{\sqrt{1 + x^2}} ds = \int_{C_1} \frac{x + y^2}{\sqrt{1 + x^2}} ds + \int_{C_2} \frac{x + y^2}{\sqrt{1 + x^2}} ds = \int_0^{1/2} \frac{1 + t^2}{\sqrt{1 + (1)^2}} \sqrt{0^2 + 1^2} dt \quad + \quad \int_0^1 \frac{(1-t) + \left[(1-t)^2/2\right]^2}{\sqrt{1 + (1-t)^2}} \sqrt{1 + (t-1)^2} dt$$

ANSWER 29, CONTINUED...

Please note that $(1-t)^2=(t-1)^2$ ("FOIL" each and see!) So, the second integral can be reduced to lower terms, i.e., the denominator "cancels" with its equivalent numerator factor.

$$=\frac{1}{\sqrt{2}}\int_0^{1/2}\left(1+t^2\right)dt+\int_0^1\left((1-t)+\frac{(1-t)^4}{4}dt\right)=\frac{1}{\sqrt{2}}\left[t+\frac{t^3}{3}\right]_0^{1/2}+\left[t-\frac{t^2}{2}-\frac{(1-t)^5}{20}\right]_0^1$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{2}+\frac{(1/2)^2}{3}\right)+\left[1-\frac{1}{2}-0\right]-\left[0-\frac{1}{20}\right]$$

$$=\frac{13}{24\sqrt{2}}+\frac{11}{20}$$

30. **Given the field** $\mathbf{F}=\left\langle 2x,-y^2,-\frac{4}{1+z^2}\right\rangle$...

(a) **Show that the field is conservative.**

Because we have $\frac{\partial P}{\partial y}=\frac{\partial N}{\partial z}=0,\ \frac{\partial P}{\partial x}=\frac{\partial M}{\partial z}=0,\ \frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}=0$, the field is conservative.

This means that there exists a potential function, $f(x,y,z)$ for which $\mathbf{F}$ is its gradient. In other words, $\nabla f(x,y,z)=\left\langle f_x,f_y,f_z\right\rangle=\mathbf{F}(x,y,z)$.

We find this potential function by integrating three times, and summing the results (but disregarding the "duplicates"):

$$f(x,y,z)=\int M dx=\int 2x dx=x^2+g(y,z)$$

$$=\int N dy=\int(-y^2)dy=-\frac{y^3}{3}+h(x,z)$$

$$=\int P dz=\int-\frac{4}{1+z^2}dz=-4\arctan z+f(x,y)$$

So, we have that:

$$f(x,y,z)=x^2-\frac{y^3}{3}-4\arctan z+C$$

(answer)

(b) **Evaluate** $\int_{(0,0,0)}^{(3,3,1)} 2x dx-y^2dy-\frac{4}{1+z^2}dz$.

We use the Fundamental Theorem of Line Integrals. We already proved the field was conservative and found its potential function. We use this now:

$$\int_{(0,0,0)}^{(3,3,1)} 2x dx-y^2dy-\frac{4}{1+z^2}dz=\left[f(x,y,z)\right]_{(0,0,0)}^{(3,3,1)}=\left[x^2-\frac{y^3}{3}-4\arctan z\right]_{(0,0,0)}^{(3,3,1)}=\left(3^2-\frac{3^3}{3}-4\arctan 1\right)-(0)$$

$$=9-9-4\left(\frac{\pi}{4}\right)=-\pi \quad \textit{(answer)}$$

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

31. Find the work done by the field $\mathbf{F} = \left\langle 2\cos y, \frac{1}{y} - 2x\sin y \right\rangle$ on the object that follows a path from the point (2, 1), to the point (2, 2), and then to the point $\left(1, \frac{\pi}{2}\right)$.

The force field is conservative since $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = -2\sin y$.

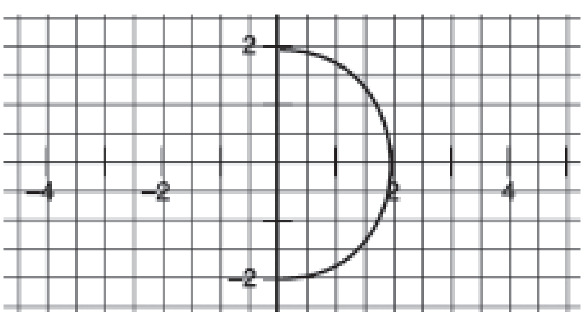
This means that the work done is independent of path and we can use the Fundamental Theorem of Line Integrals. In other words, there is no need to parameterize the curves and we only care about the starting and ending points! We find this potential function by integrating two times, and summing the results (but disregarding the "duplicates"):

Next, we evaluate the line integral using the Fundamental Theorem of Line Integrals:

Work = $\int_C \mathbf{F}\cdot d\mathbf{r} = \ln|y| + 2x\cos y\Big]_{(2,1)}^{(1,\pi/2)} = [\ln(\pi/2) + 2(1)\cos(\pi/2)] - [\ln(1) + 2(2)\cos(1)] = \ln(\pi/2) - 4\cos(1)$ *(answer)*

32. Use Green's Theorem to evaluate the line integral $\int_C (x - y^3)dx + x^3 dy$, where *C* is the right half of a circle of radius 2, $x^2 + y^2 = 4$.

Since the path is closed, we may use Green's Theorem. First, sketch the path in order to determine the region of integration:



First, find $\frac{\partial N}{\partial x}$ *and* $\frac{\partial M}{\partial y}$,

which are

$$\frac{\partial N}{\partial x} = 3x^2 \text{ and } \frac{\partial M}{\partial y} = -3y^2.$$

Next, apply Green's Theorem:

$$\int_C (x - y^3)dx - x^3 dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(3x^2 - \left(-3y^2\right)\right) dydx$$

This double integral will be easier to evaluate if we convert to polar coordinates.

Please note that the range for the angle θ is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and *not* $\frac{3\pi}{2} \le \theta \le \frac{\pi}{2}$. If you do this, you will lose *lots* of points due to the fact that the lower limit is actually a number larger than the upper limit—so the double inequality is in fact *meaningless!*

So now, we have that:

$$\int_{-\pi/2}^{\pi/2}\int_0^2 3(x^2+y^2)r\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^2 3(r^2)r\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^2 3(r^3)\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\left[\frac{3}{4}r^4\right]_0^2 d\theta=\int_{-\pi/2}^{\pi/2}\left[\frac{3}{4}2^4\right]d\theta$$

$$=12\int_{-\pi/2}^{\pi/2}d\theta=12[\theta]_{-\pi/2}^{\pi/2}=12\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=12\pi$$

33. Evaluate the surface integral $\iint_S y\,dS$ if S is the part of the plane $z=6-3x-2y$ in the first octant.

We have that $g(x, y) = z = 6 - 3x - 2y$.

We find $dS=\sqrt{1+[g_x]^2+[g_y]^2}=\sqrt{1+(-3)^2+(-2)^2}=\sqrt{14}$.

The region of integration in the xy-plane is the projection of the surface (the plane: $(z = 6 - 3x - 2y)$) onto the xy-plane. Let $z = 0$ and sketch the region if you need to:

$$\iint_S y\,dS=\iint_R y\sqrt{14}\,dA=\int_0^2\int_0^{(6-3x)/2} y\sqrt{14}\,dy\,dx=\sqrt{14}\int_0^2\left[\frac{y^2}{2}\right]_0^{(6-3x)/2}dx=\sqrt{14}\int_0^2\frac{(6-3x)^2}{8}dx$$

Use u-substitution, where $u=6-3x$, $du=-3dx$:

$$\frac{\sqrt{14}}{8}\left(-\frac{1}{3}\right)\int_0^2(6-3x)^2(-3)dx=-\frac{\sqrt{14}}{24}\left[\frac{u^3}{3}\right]=-\frac{\sqrt{14}}{72}\left[(6-3x)^3\right]_0^2=-\frac{\sqrt{14}}{72}\left[(6-3(2))^3-(6-3(0))^3\right]=-\frac{\sqrt{14}}{72}(-6^3)=3\sqrt{14}$$

34. Find the flux integral if $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$ if $\mathbf{F}=\langle x, y, z\rangle$ where S is the surface $z = 1 - x^2 - y^2$ above the xy-plane.

Use the fact that $\iint_S \mathbf{F}\cdot\mathbf{N}dS=\iint_R F\cdot\langle -g_x,-g_y,1\rangle dA$, where $g(x, y) = z = 1 - x^2 - y^2$.

So, we get the vector $\langle -g_x,-g_y,1\rangle=\langle -(-2x),-(-2y),1\rangle=\langle 2x,2y,1\rangle$. The region of integration will be in the xy-plane. We let $z = 0$ in the equation $z = 1 - x^2 - y^2$, and find that we have a circle of radius 1. Now, our double integral is of the form:

$$\iint_R F\cdot\langle -g_x,-g_y,1\rangle dA=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\langle x,y,z\rangle\cdot\langle 2x,2y,1\rangle dy\,dx=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2x^2+2y^2+z)dy\,dx$$

$$=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(2x^2+2y^2+(1-x^2-y^2))dy\,dx$$

We replaced "z" with the equation for the surface $z = 1 - x^2 - y^2$ in that last step. This is because you are only allowed to have two different variables for double integrals, not three variables! Also, we will convert to polar coordinates to make the integral easier to evaluate:

$$\int_0^{2\pi}\int_0^1(2r^2+1-r^2)r\,dr\,d\theta=\int_0^{2\pi}\int_0^1(r^3+r)\,dr\,d\theta=\int_0^{2\pi}\left[\frac{r^4}{4}+\frac{r^2}{2}\right]_0^1 d\theta=\frac{3}{4}\int_0^{2\pi}d\theta=\frac{3}{4}(2\pi)=\frac{3\pi}{2}\quad\textit{(answer)}$$

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

35. Let Q be the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$, and let $\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$. Source, sink, or neither?

First, we'll find the divergence of **F**:

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2, y^2, z^2 \rangle = \frac{\partial[x^2]}{\partial x} + \frac{\partial[y^2]}{\partial y} + \frac{\partial[z^2]}{\partial z} = 2x + 2y + 2z$$

Next, we apply the Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q div\mathbf{F} dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x + 2y + 2z)dzdydx = \int_{-1}^{1}\int_{-1}^{1}\left[2xz + 2yz + z^2\right]_{-1}^{1} dydx$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[2x(1) + 2y(1) + (1)^2\right] - \left[2x(-1) + 2y(-1) + (-1)^2\right]dydx = \int_{-1}^{1}\int_{-1}^{1}(4x + 4y)dydx = \int_{-1}^{1}\left[4xy + 2y^2\right]_{-1}^{1} dx$$

$$= \int_{-1}^{1}\left[4x(1) + 2(1)^2\right] - \left[4x(-1) + 2(-1)^2\right]dx = \int_{-1}^{1} 8xdx = \left[4x^2\right]_{-1}^{1} = 4(1)^2 - 4(-1)^2 = 0 \quad (answer)$$

Since the answer is zero, the outward flux of the field **F** through the surface is *neither* a source or a sink (i.e., incompressible).

36. Let Q be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$ and calculate the outward flux of F through S where S is the surface of Q and $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$. Source, sink, or neither?

The divergence of **F** is: $div\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{\partial[x]}{\partial x} + \frac{\partial[y]}{\partial y} + \frac{\partial[z]}{\partial z} = 1 + 1 + 1 = 3$

We now apply the Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q div\mathbf{F} dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1} 3dzdydx$

We will convert to cylindrical coordinates to make it easier to integrate:

$$= \int_0^{2\pi}\int_0^1\int_0^1 3dzrdrd\theta = \int_0^{2\pi}\int_0^1 3[z]_0^1 rdrd\theta = 3\int_0^{2\pi}\int_0^1 rdrd\theta = 3\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{3}{2}\int_0^{2\pi} d\theta = \frac{3}{2}[\theta]_0^{2\pi} = \frac{3}{2}(2\pi) = 3\pi \; (answer)$$

Since the answer is positive, we have a *source*.

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

37. Let Q be the region bounded above by the sphere $x^2 + y^2 + z^2 = 9$ and below by the plane $z = 0$ in the first quadrant. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{N} dS$ and find the outward flux of F through S, where S is the surface of the solid and $\mathbf{F}(x, y, z) = \langle xy, 4x, 2y \rangle$. Source, sink, or neither?

First we get the divergence of $\mathbf{F}$: $div\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = \frac{\partial [xy]}{\partial x} + \frac{\partial [4x]}{\partial y} + \frac{\partial [2y]}{\partial z} = y$

Next, we apply the theorem: $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iiint_Q div\mathbf{F} dV = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} y dz dy dx$

We will convert to spherical coordinates to make it easier to integrate. Keep in mind that the "drop-down" angle, ϕ, ranges from $0 \le \phi \le \frac{\pi}{2}$, since we are bounded below by the xy-plane:

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 (\rho \sin\phi \sin\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 (\rho^3 \sin^2\phi \sin\theta) d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \sin^2\phi \sin\theta \left[\frac{\rho^4}{4}\right]_0^3 d\phi \, d\theta$$

$$= \frac{81}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin^2\phi \sin\theta \, d\phi \, d\theta = \frac{81}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\frac{1 - \cos(2\phi)}{2}\right] d\phi \, d\theta = \frac{81}{4}\left(\frac{1}{2}\right) \int_0^{\pi/2} \sin\theta \left[\phi - \frac{1}{2}\sin(2\phi)\right]_0^{\pi/2} d\theta$$

$$= \frac{81}{8} \int_0^{\pi/2} \sin\theta \left(\frac{\pi}{2}\right) d\theta = \frac{81\pi}{16}[-\cos\theta]_0^{\pi/2} = \frac{81\pi}{16}\left[-\cos\frac{\pi}{2} - (-\cos 0)\right] = \frac{81\pi}{16}(0 + 1) = \frac{81\pi}{16}$$

SOURCE *(answer)*

38. Find the curl of the vector field $\mathbf{F}(x, y, z) = \langle z^2, -x^2, y \rangle$. Is the field conservative?

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -x^2 & y \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2 & y \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ z^2 & y \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ z^2 & -x^2 \end{vmatrix}\mathbf{k}$$

$$= \left(\frac{\partial[y]}{\partial y} - \frac{\partial[-x^2]}{\partial z}\right)\mathbf{i} - \left(\frac{\partial[y]}{\partial x} - \frac{\partial[z^2]}{\partial z}\right)\mathbf{j} + \left(\frac{\partial[-x^2]}{\partial x} - \frac{\partial[z^2]}{\partial y}\right)\mathbf{k} = (1-0)\mathbf{i} - (0-2z)\mathbf{j} + (-2x-0)\mathbf{k} = \langle 1, 2z, -2x \rangle$$

Since the curl is not equal to the zero vector, the field is *not* conservative. The curl of a conservative vector field is always equal to the zero vector.

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

39. Use Stokes's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=\langle z, 2x, 2y\rangle$ and *S* is the surface of the paraboloid (oriented upward) of $z=4-x^2-y^2$, $z\ge 0$, and *C* is its boundary.

First, we have that $g(x,y)=z=4-x^2-y^2$.

So, $\mathbf{N}dS=\langle -g_x, -g_y, 1\rangle dA=\langle -(-2x),-(-2y),1\rangle dA=\langle 2x, 2y, 1\rangle dA$.

Next, we find the curl of the field:

$$curl\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & 2x & 2y\end{vmatrix}=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x & 2y\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ z & 2y\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ z & 2x\end{vmatrix}\mathbf{k}$$

$$=\left(\frac{\partial[2y]}{\partial y}-\frac{\partial[2x]}{\partial z}\right)\mathbf{i}-\left(\frac{\partial[2y]}{\partial x}-\frac{\partial[z]}{\partial z}\right)\mathbf{j}+\left(\frac{\partial[2x]}{\partial x}-\frac{\partial[z]}{\partial y}\right)\mathbf{k}=(2-0)\mathbf{i}-(0-1)\mathbf{j}+(2-0)\mathbf{k}=\langle 2,1,2\rangle$$

Next, we apply Stokes' Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S curl\mathbf{F}\cdot\mathbf{N}dS=\iint_R \langle 2,1,2\rangle\cdot\langle 2x, 2y, 1\rangle dA=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4x+2y+2)\,dydx$$

Convert to polar:

$$=\int_0^{2\pi}\int_0^{2}(4r\cos\theta+2r\sin\theta+2)\,rdrd\theta=\int_0^{2\pi}\int_0^{2}(4r^2\cos\theta+2r^2\sin\theta+2r)\,drd\theta=\int_0^{2\pi}\left[\frac{4r^3}{3}\cos\theta+\frac{2r^3}{3}\sin\theta+r^2\right]_0^2 d\theta$$

$$=\int_0^{2\pi}\left[\frac{4(2)^3}{3}\cos\theta+\frac{2(2)^3}{3}\sin\theta+(2)^2\right]_0^2 d\theta=\int_0^{2\pi}\left[\frac{32}{3}\cos\theta+\frac{16}{3}\sin\theta+4\right]_0^2 d\theta=\left[\frac{32}{3}\sin\theta-\frac{16}{3}\cos\theta+4\theta\right]_0^{2\pi}=8\pi$$

CALCULUS III — Practice FINAL EXAM KEY, CONTINUED

40. Limits: Each of the following problems requires knowledge of limits.

(a) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{t\to 0}\left\langle \frac{\sin t - t}{t^3}, \frac{3t}{t-1}, \arctan\left(\frac{e^{t^2}}{2t^2}\right)\right\rangle$$

Always try to start all limit problems using direct substitution. So, after substituting $t = 0$ into the first component, we have: $\frac{\sin 0 - 0}{(0)^3} = \frac{0}{0}$. Remember that $\frac{0}{0}$ is "indeterminate" which means it is anybody's guess what it represents. Please do not say that $\frac{0}{0}$ is equal to 0. Instead, know that anytime you run into the quotient $\frac{0}{0}$, you are probably going to use L'Hôpital's Rule if the expression is single-valued as this one is. This indeterminate form is one of the kinds where it is permissible to use L'Hôpital's Rule. After taking the derivative of the numerator and dividing it by the derivative of the denominator, we will again use direct substitution:

$$\lim_{t\to 0}\frac{\sin t - t}{t^3} = \lim_{t\to 0}\frac{\cos t - 1}{3t^2} = \frac{\cos 0 - 1}{3(0)^2} = \frac{1-1}{0} = \frac{0}{0}$$

We need to apply L'Hôpital's Rule again!

$$\lim_{t\to 0}\frac{-\sin t}{6t} = \frac{0}{0}$$

Apply L'Hôpital's Rule one more time:

$$\lim_{t\to 0}\frac{-\sin t}{6t} = \lim_{t\to 0}\frac{-\cos t}{6} = -\frac{1}{6}$$

For the second component, we use direct substitution: So, after substituting $t = 0$ into the second component, we have:

$$\frac{3(0)}{0-1} = \frac{0}{-1} = 0$$

For the third component, we use direct substitution, but find we need to take a look at the argument of the inverse tangent a bit more closely. So, after substituting $t = 0$ into the third component, we have:

$$\arctan\left(\frac{e^0}{2(0)^2}\right) = \arctan\left(\frac{1}{0}\right)$$

ANSWER 40A, CONTINUED...

We find that the limit of $\lim_{t\to 0}\frac{e^{t^2}}{2t^2}=+\infty$. Then:

$$\arctan(\infty)=\frac{\pi}{2}$$

The ***final answer*** must be a vector, which is:

$$\lim_{t\to 0}\mathbf{r}(t)=\left\langle -\frac{1}{6},\, 0,\, \frac{\pi}{2}\right\rangle$$

(b) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{t\to\infty}\left\langle e^{-t},\ \frac{-t^2-2}{3t^2+5},\ \frac{t^2}{\ln t}\right\rangle$$

After applying direct substitution to the first component, we have the limit as: $e^{-\infty}=0$. So, the limit exists for the first component. Next, we apply direct substitution to the second component to obtain:

$$\lim_{t\to\infty}\frac{-t^2-2}{3t^2+5}=\frac{-\infty^2-2}{3\infty^2+5}=\frac{-\infty}{\infty}$$

This is an indeterminate form where it is permissible to apply L'Hôpital's Rule:

$$\lim_{t\to\infty}\frac{-t^2-2}{3t^2+5}=\lim_{t\to\infty}\frac{-2t}{6t}=\frac{-2}{6}=-\frac{1}{3}$$

So, the limit exists for the second component. Apply direct substitution to the last component $\lim_{t\to\infty}\frac{t^2}{\ln t}=\frac{\infty^2}{\ln(\infty)}=\frac{\infty}{\infty}$.

At this point, you may use repeated applications of L'Hôpital's Rule or use the fact that polynomials tend to infinity "faster" than logarithms to get that this limit does not exist. Since the limit does not exist for this last component, the final answer for this problem is:

"The limit does not exist."

(c) Find the limit at the boundary point (if it exists):

$$\lim_{(x,y)\to(3,2)}\frac{6x^2-13xy+6y^2}{6x^2-xy-12y^2}$$

We always try and do direct substitution first, as we have done for all limit problems in your first year of calculus classes. So, we do that here:

$$\lim_{(x,y)\to(3,2)}\frac{6x^2-13xy+6y^2}{6x^2-xy-12y^2}=\frac{6(3)^2-13(3)(2)+6(2)^2}{6(3)^2-(3)(2)-12(3)^2}=\frac{54-78+24}{54-6-12(2)^2}=\frac{0}{0}$$

This result is an indeterminate form to which we used to be able to apply L'Hôpital's Rule—but cannot, as this is a function of two variables. Instead, let's try and simplify the expression. It turns out that both the numerator and denominator are factorable trinomials and we can reduce the quotient to lowest terms before re-evaluating! In other words, we will use the Replacement Theorem that you learned in Calculus I. I will use the Master Product Method to factor the trinomials. I will require all students to be able to show me the steps to do this on their exams in order to receive full credit. For the numerator, the "Key Number" for the trinomial is the product of the lead coefficients of the two variables: (6)(6) = 36.

Factors of 36 that add up to the middle coefficient (which is -13) are: -9 and -4.

We rewrite the trinomial such that the middle term is split into two using these two factors, as shown below. For the denominator trinomial, the "Key Number" is the product of the lead coefficients of the two variables: (6)(-12) = -72. Factors of -72 that add up to the middle coefficient (which is -1) are: 8 and -9.

We rewrite each trinomial such that the middle term is split into two using these two factors:

$$\lim_{(x,y)\to(3,2)} \frac{6x^2-13xy+6y^2}{6x^2-xy-12y^2} = \lim_{(x,y)\to(3,2)} \frac{6x^2-9xy-4xy+6y^2}{6x^2-9xy+8xy-10y^2}$$

Now, we factor by grouping, and then apply direct substitution to the reduced quotient:

$$\lim_{(x,y)\to(3,2)} \frac{\left(6x^2-9xy\right)+\left(-4xy+6y^2\right)}{\left(6x^2-9xy\right)+\left(8xy-10y^2\right)} = \lim_{(x,y)\to(3,2)} \frac{3x(2x-3y)-2y(2x-3y)}{3x(2x-3y)+4y(2x-3y)}$$

$$\lim_{(x,y)\to(3,2)} \frac{(3x-2y)(2x-3y)}{(2x-3y)(3x+4y)} = \lim_{(x,y)\to(3,2)} \frac{3x-2y}{3x+4y} = \frac{3(3)-2(2)}{3(3)+4(2)} = \frac{5}{17}$$

Therefore, the limit exists and is equal to: $\frac{5}{17}$

(d) Find the limit (if it exists). If it does not exist, so state.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{\sin(x^2+y^2+z^2)}$$

After direct substitution, we have:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{\sin(x^2+y^2+z^2)} = \frac{0^2+0^2+0^2}{\sin\left(0^2+0^2+0^2\right)} = \frac{0}{0}$$

ANSWER 49D, CONTINUED...

This result is an indeterminate form to which we used to be able to apply L'Hôpital's Rule—but cannot as this is a function of two variables. We also cannot use the Replacement Theorem as we did in the last problem because this quotient is already completely reduced to lowest terms. Instead, we will use a Theorem you used in Calculus I: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, where the angle θ here is the same as: $\theta = x^2+y^2+z^2$.

So, the limit exists and is equal to 1.

(e) **Find the limit (if it exists). If it does not exist, so state.** ***Hint:*** **Convert to polar coordinates and use the fact that $(x,y)\to(0,0)$ means the same as $r\to 0$ along all paths in the domain to the point (0, 0):**

$$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2+x^2y^2}{x^2+y^2}=\lim_{r\to 0}\frac{r^2\cos^2\theta+r^2\sin^2\theta+\left(r^2\cos^2\theta\right)\left(r^2\sin^2\theta\right)}{r^2\cos^2\theta+r^2\sin^2\theta}$$

$$=\lim_{r\to 0}\frac{r^2(\cos^2\theta+\sin^2\theta)+r^4\cos^2\theta\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)}=\lim_{r\to 0}\frac{r^2(1+r^2\cos^2\theta\sin^2\theta)}{r^2}=\lim_{r\to 0}\left(1+r^2\cos^2\theta\sin^2\theta\right)$$

$$=\ 1+0=1$$

So, the limit exists and is equal to 1

(f) **True or False:**

The improper integral $\int_0^\infty\int_0^x\left(\frac{1}{x^2+1}\right)\left(\frac{1}{y^2+1}\right)dydx$ converges to the value: $\frac{\pi^2}{4}$

The answer is: FALSE. Here is the work to show the statement is invalid:

$$\int_0^\infty\int_0^x\left(\frac{1}{x^2+1}\right)\left(\frac{1}{y^2+1}\right)dydx=\int_0^\infty\frac{1}{x^2+1}\left[\arctan y\right]_0^x\,dx=\int_0^\infty\frac{\arctan x}{x^2+1}dx$$

Next, we use the Power rule with u-substitution, where $u=\arctan x,\ du=\frac{1}{x^2+1}dx$:

$$\int_0^\infty\frac{\arctan x}{x^2+1}dx=\int u\,du=\frac{u^2}{2}\Rightarrow\left[\frac{(\arctan x)^2}{2}\right]_0^\infty=\frac{(\arctan\infty)^2}{2}-\frac{(\arctan 0)^2}{2}=\frac{(\pi/2)^2}{2}-0=\frac{\pi^2}{8}$$

So, the improper integral converges to $\frac{\pi^2}{8}$, and not $\frac{\pi^2}{4}$.